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TECHNICAL NOTE

No. 1412

VOLTERRA'S SOLUTION OF THE WAVE EQUATION
AS APPLIED TO THREE-DIMENSIONAL
SUPERSONIC AIRFOIL PROBLEMS

By Max. A. Heaslet, Harvard Lomax, and
Arthur L. Jones

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SUMMARY

A surface integral is developed which yields solutions of the linearized partial differential equation for supersonic flow. These solutions satisfy boundary conditions arising in wing theory. Particular applications of this general method are made, using acceleration potentials, to flat surfaces and to uniformly loaded lifting surfaces. Rectangular and trapezoidal plan forms are considered along with triangular forms adaptable to swept-forward and swept-back wings. The case of the triangular plan form in sideslip is also included. Emphasis is placed on the systematic application of the method to the lifting surfaces considered and on the possibility of further application.

INTRODUCTION

The study of supersonic lifting surfaces has been limited, until very recent times, to the work of relatively few theorists. The introduction of new propulsive systems has, however, increased the possible speed range of aircraft to such a degree that a great deal of emphasis has been placed on extending theoretical knowledge and techniques to cover controlled-flight velocities well beyond those in the subsonic regime. As a result of this added emphasis, within the past two or three years a number of important developments and results have been reported which add materially to the main body of supersonic theory and included among these publications are several papers on wing theory for both the lifting and nonlifting cases. Most of these solutions, however, have been limited to the field of conical flow, while a generalized theory, which would fit any boundary condition likely to be met in problems of wing theory, has been lacking.

In the present report advantage has been taken of the direct

analogy which exists between the linearized partial differential equation for supersonic flow in three dimensions and the two-dimensional wave equation of mathematical physics. As a result of this correspondence, solutions which have been given for the wave equation are shown to be applicable to the type of boundary condition encountered in wing problems. The first section of the report is devoted to the development of the mathematical equation for the potential of the supersonic flow field. The application of this expression to a number of examples in supersonic lifting-surface theory illustrates the usefulness of this method of solution. In the first of these examples the loadings over the given plan forms are assumed to be constant. The results obtained for such cases appear at first to be somewhat academic since undesirable twist and camber occur over portions of the resultant surfaces. From the constantly loaded surfaces, however, it is possible to develop surfaces having arbitrary load distributions. Imposing the condition that the final lifting surface shall be a flat plate leads to the solution of an integral equation in every case considered. The results obtained, for some of the plan forms considered, have been developed elsewhere but not with the unification of method attained here. New configurations are also included among the examples given. The methods shown are applicable to a large class of unsolved problems of immediate interest.

LIST OF SYMBOLS

(In order of their appearance)

Φ	velocity potential
x, y, z	Cartesian coordinates
a	local velocity of sound
Ω, σ	variable representing either the acceleration potential, the velocity potential, or any of the three perturbation velocity components
M	free-stream Mach numbers
X, Y, Z	transformed coordinates (See equation (3).)
S	surface enclosing volume V
V	volume

n_1, n_2, n_3	direction cosines of normals to surface S
v_1, v_2, v_3	direction cosines of conormal v to surface S
τ	surface on which boundary conditions are given
$P(X,Y,Z)$	point at which value of Ω is to be determined
Γ	Mach forecone from point $P(X,Y,Z)$
λ	surface cutting axis of forecone Γ normally
K	cylinder enclosing axis of forecone Γ
u,v,w	perturbation velocities in direction of X -, Y -, and Z -axes, respectively
β	$\sqrt{M^2-1}$
ϕ	acceleration potential
ϕ_u	value of acceleration potential on upper side of lifting surface
ϕ_l	value of acceleration potential on lower side of lifting surface
ρ_0	density in the free stream
p_l	static pressure on lower side of lifting surface
p_u	static pressure on upper side of lifting surface
C_0	constant value of discontinuity in ϕ over uniformly loaded lifting surface
V_0	free-stream velocity
μ	Mach angle of the free stream $\left(\mu = \sin^{-1} \frac{1}{M}\right)$
b	span of wing
c	chord of wing
η	conical flow coordinate (See equations (27) and (30).)
δ	angle measured from X -axis

ω	$1 - \frac{c}{x}$
θ	$\beta \tan \delta$
α	angle of attack, radians
Δp	pressure differential ($p_l - p_u$)
q	free-stream dynamic pressure ($\frac{1}{2} \rho_o V_o^2$)
$C(\theta)$	load distribution
L	lift of wing
S_o	area of wing
C_L	lift coefficient (L/qS_o)
A	aspect ratio (b^2/S_o)
$\left. \begin{array}{l} \text{sn } (u, k), \\ \text{cn } (u, k), \\ \text{dn } (u, k) \end{array} \right\}$	Jacobi's elliptic functions of argument u and modulus k
$F(u, k)$	incomplete elliptic integral of first kind with argument u and modulus k
K, K'	complete elliptic integrals of first kind with modulus k and $\sqrt{1-k^2}$, respectively
$E(u, k)$	incomplete elliptic integral of second kind with argument u and modulus k
E, E'	complete elliptic integrals of second kind with modulus k and $\sqrt{1-k^2}$, respectively
$\Pi(u, \gamma)$	incomplete elliptic integral of third kind with argument u , parameter γ , and modulus k
$\Theta(u)$	Jacobi's theta function
Δ	$\frac{1}{2}$ vertex angle of triangular wing
Λ	angle of sideslip
H_1, H_2	functions introduced in equations (105) and (106)

THEORY

Linearization of Differential Equation for Compressible Flow

The quasi-linear (i.e., linear in the derivatives of highest order) differential equation for the velocity potential Φ in the case of compressible fluid flow in three dimensions, is expressible in the form

$$\begin{aligned} \Phi_{xx} \left(1 - \frac{\Phi_x^2}{a^2} \right) + \Phi_{yy} \left(1 - \frac{\Phi_y^2}{a^2} \right) + \Phi_{zz} \left(1 - \frac{\Phi_z^2}{a^2} \right) \\ - 2 \Phi_{yz} \frac{\Phi_y \Phi_z}{a^2} - 2 \Phi_{zx} \frac{\Phi_z \Phi_x}{a^2} - 2 \Phi_{xy} \frac{\Phi_x \Phi_y}{a^2} = 0 \end{aligned} \quad (1)$$

where a represents the local velocity of sound in the medium and Cartesian coordinates are used. Under the assumptions of small perturbation theory (references 1 and 2), this equation is modified so that it is linear in form and consequently more amenable to mathematical analysis. Denoting by the variable Ω either the acceleration potential, the velocity potential, or any of the three perturbation velocity components, the linearized expression for equation (1) is

$$(1 - M^2) \Omega_{xx} + \Omega_{yy} + \Omega_{zz} = 0 \quad (2)$$

where M is the Mach number of the free stream and thus equal to the ratio of free-stream velocity and the corresponding speed of sound.

By means of the affine transformation

$$\left. \begin{aligned} X &= x \\ Y &= \sqrt{\pm(1 - M^2)} y \\ Z &= \sqrt{\pm(1 - M^2)} z \end{aligned} \right\} \quad (3)$$

equation (2) can be put into standard forms. (The term affine refers to an arbitrary linear transformation of coordinates.) Thus, when $M < 1$ the plus signs are chosen in the radicals of equation (3) and equation (2) becomes

$$\Omega_{XX} + \Omega_{YY} + \Omega_{ZZ} = 0 \quad (4)$$

while for $M > 1$ the minus signs are used and, as a consequence,

$$\Omega_{XX} - \Omega_{YY} - \Omega_{ZZ} = 0 \quad (5)$$

For the case of subsonic flow ($M < 1$) the linearized equation is thereby reduced to the well known Laplace equation in three-dimensions. Similarly, in supersonic flow ($M > 1$) equation (2) is again reduced to classical type with the replacement of the space coordinate X by a time variable T to give the two-dimensional wave equation of mathematical physics. The linearization of the general differential equation for compressible fluid flow therefore makes available, in both subsonic and supersonic studies, the results of the extensive work carried out in previous research on problems related to equations (4) and (5).

Application of Green's Theorem to Linearized Compressible Flow Equation

Methods of solution for partial differential equations of the type considered here may be classified into two principal categories: methods which express the solutions in terms of orthogonal functions and methods which are based on the use of Green's theorem. Volterra's solution, discussions of which may be found in references 3, 4, and 5, applies the latter approach to the two-dimensional wave equation and, as a consequence, his results may be adapted to the study of supersonic flow and specific solutions of equation (5). Let

$$L(\Omega) \equiv \Omega_{XX} - \Omega_{YY} - \Omega_{ZZ}$$

The analytic form of Green's theorem for equation (5), relating a volume integral over the region V to a surface integral over the surface S enclosing V , may be written in the form

$$\iiint_V [\sigma L(\Omega) - \Omega L(\sigma)] dv = - \iint_S [\sigma D_n \Omega - \Omega D_n \sigma] dS$$

where σ , Ω are any two functions which, together with their first and second derivatives, are finite and single valued throughout the region considered, and

$$D_n \Omega = n_1 \frac{\partial \Omega}{\partial X} - n_2 \frac{\partial \Omega}{\partial Y} - n_3 \frac{\partial \Omega}{\partial Z}$$

where n_1, n_2, n_3 are direction cosines of normals to the surface S .

The expression for $D_n \Omega$ is, of course, a directional derivative. The corresponding term appearing in Green's theorem for Laplace's differential equation (incompressible fluid flow) is precisely the directional derivative along the normal to the surface S . The analogy between the two terms prompts the introduction of the so-called conormal to S with direction cosines v_1, v_2, v_3 defined as

$$v_1 = -n_1, v_2 = n_2, v_3 = n_3$$

The geometrical connection between the normal and the conormal is indicated in figure 1; the angles between the lines and the Y - and Z -axes remain respectively equal, while the angles between the lines and the X -axis are supplementary. It follows, in particular, that if the surface S is the XY plane the two lines are coincident; if S is a cone with semivertex angle equal to 45° and axis parallel to the X -axis, then the conormal at any point lies along the surface S .

It is now possible to write

$$-D_n \Omega = v_1 \frac{\partial \Omega}{\partial X} + v_2 \frac{\partial \Omega}{\partial Y} + v_3 \frac{\partial \Omega}{\partial Z} = \frac{\partial \Omega}{\partial v} \quad (6)$$

and the surface-volume relation becomes

$$\iiint_V [\sigma L(\Omega) - \Omega L(\sigma)] dV = \iint_S \left[\sigma \frac{\partial \Omega}{\partial v} - \Omega \frac{\partial \sigma}{\partial v} \right] dS \quad (7)$$

If Ω and σ are chosen so as to satisfy equation (5) throughout the region V , then equation (7) reduces to the form

$$\iint_S \sigma \frac{\partial \Omega}{\partial v} dS = \iint_S \Omega \frac{\partial \sigma}{\partial v} dS \quad (8)$$

The form of equation (8) is a direct analogue to results obtainable for functions satisfying Laplace's equation. (See, for instance, reference 6, page 46.) The use of the conformal produces this symbolic equivalence.

Volterra's Method for Three-Dimensional Wave Equation

Consider now a surface τ which, for the purposes of this report, may be thought of as being coincident with the XY plane and parallel to the air flow which is in the direction of the positive X-axis. Two such surfaces are represented by the darkened areas in figure 2. It is desired to determine the value of Ω at the point $P(X,Y,Z)$ from a knowledge of the boundary conditions given on τ . The solution to such a problem is immediately suggested by equation (8) since that equation requires only the knowledge of Ω and $\frac{\partial \Omega}{\partial v}$ along a surface enclosing a given volume, together with the knowledge of some particular solution σ to the wave equation valid everywhere within the enclosed volume. Further, it is physically evident that in order that it contribute to the solution the enclosed volume should be within the forecone from the point P and also within the envelopes of the aftercones from the foremost disturbances from P . Referring to figure 2(a), this would mean the volume bounded by the forecone Γ and the wedge λ springing from the leading edge of τ ; and in figure 2(b), the volume bounded by the forecone Γ and the aftercone λ springing from the apex of the surface τ , where in each case τ represents the portion of the disturbance region affecting the value of Ω at P . Since for the boundary value problems involved the surface τ remains in the XY plane, equation (8) must be applied to all three surfaces λ , Γ , and τ .

Since there is no way of determining Ω and $\frac{\partial \Omega}{\partial v}$ along Γ the attempted solution must fail unless the particular solution σ and its derivative with respect to the conormal vanish everywhere on Γ . But this is in fact the essential part of Volterra's method of solution. Thus the proper choice of σ is

$$\sigma = \text{arc cosh} \frac{X-X_1}{\sqrt{(Y-Y_1)^2 + (Z-Z_1)^2}}$$

(This relation, incidentally, is the indefinite integral of the fundamental solution representing a supersonic source in three dimensions $[(X-X_1)^2 - (Y-Y_1)^2 - (Z-Z_1)^2]^{-1/2}$.) The value of σ is equal to zero on the forecone Γ since the equation of this cone is

$$(X-X_1)^2 - (Y-Y_1)^2 - (Z-Z_1)^2 = 0$$

and further, since the conormal is always directed along the forecone, $\frac{\partial \sigma}{\partial v}$ is the gradient of σ along Γ and is also zero.

Equation (8) now provides an equality for the distribution of Ω and $\frac{\partial \Omega}{\partial v}$ over λ and τ provided Ω and σ satisfy equation (5) throughout the enclosed volume mentioned. However, although σ satisfies equation (5) everywhere in the enclosed volume opposite τ from P—under the XY plane in figure 2—along the line $(Y-Y_1)^2 + (Z-Z_1)^2 = 0$ —above the XY plane in figure 2— σ is infinite and does not satisfy equation (5). If this line is excluded by means of a cylinder K of radius ϵ , with axis lying along the line $(Y-Y_1)^2 + (Z-Z_1)^2 = 0$, then equation (8) may be applied to the region outside K and yet within the space bounded by λ , τ , and Γ . In fact equation (8) can then be written

$$\iint_{\tau_1 + K + \lambda} \left(\Omega \frac{\partial \sigma}{\partial v} - \sigma \frac{\partial \Omega}{\partial v} \right) dS = 0 \quad (9)$$

where τ_1 is the portion of τ bounding the region of integration.

If $R = \sqrt{(Y-Y_1)^2 + (Z-Z_1)^2}$ and cylindrical coordinates ϵ , ψ , and $(X-X_1)$ are used, an element of area on the cylinder K is $dS = -\epsilon d\psi d(X-X_1)$, while

$$\frac{\partial \sigma}{\partial v} = \frac{\partial \sigma}{\partial R} = - \frac{(X-X_1)}{\epsilon \sqrt{(X-X_1)^2 - \epsilon^2}}$$

so that

$$\begin{aligned} & \lim_{\epsilon \rightarrow 0} \iint_K \left(\Omega \frac{\partial \sigma}{\partial v} - \sigma \frac{\partial \Omega}{\partial v} \right) dS \\ &= \lim_{\epsilon \rightarrow 0} \iint_K \frac{\epsilon \Omega (X-X_1) d\psi d(X-X_1)}{\epsilon \sqrt{(X-X_1)^2 - \epsilon^2}} - \lim_{\epsilon \rightarrow 0} \iint_K \epsilon \frac{\partial \sigma}{\partial v} \cosh \left(\frac{X-X_1}{\epsilon} \right) d\psi d(X-X_1) \\ &= \iint_K -\Omega d\psi dX_1 - \iint_K \lim_{\epsilon \rightarrow 0} \frac{\partial \Omega}{\partial v} \left(\ln - \frac{X_1-X}{\epsilon} \right) \epsilon d\psi d(X-X_1) \\ &= -2\pi \int_X^{X_\lambda} \Omega(\xi, Y, Z) d\xi \end{aligned} \quad (10)$$

Applying this result to equation (9) gives

$$-2\pi \int_{X_\lambda}^X \Omega(\xi, Y, Z) d\xi = \iint_{\tau_1 + \lambda} \left(\Omega \frac{\partial \sigma}{\partial v} - \sigma \frac{\partial \Omega}{\partial v} \right) dS \quad (11)$$

and, differentiating equation (11) with respect to X , yields

$$\Omega(X, Y, Z) = -\frac{1}{2\pi} \frac{\partial}{\partial X} \iint_{\tau_1 + \lambda} \left(\Omega \frac{\partial \sigma}{\partial v} - \sigma \frac{\partial \Omega}{\partial v} \right) dS \quad (12)$$

Procedure for Lifting Surfaces and Symmetric Wings

When the region considered is that bounded by the surface τ , Γ , and λ' , the portion of λ on the opposite side of τ from the point P , then σ is finite throughout the region and, as a direct consequence of equation (9)

$$0 = -\frac{1}{2\pi} \frac{\partial}{\partial X} \iint_{\tau_1 + \lambda'} \left(\Omega' \frac{\partial \sigma}{\partial v'} - \sigma \frac{\partial \Omega'}{\partial v'} \right) dS \quad (13)$$

where Ω' is the value of the potential function on the side of τ opposite P and v' is in the opposite direction to v . Combining equations (12) and (13) gives

$$\begin{aligned} \Omega(X, Y, Z) = & \frac{1}{2\pi} \frac{\partial}{\partial X} \iint_{\tau_1} \left(\frac{\partial \Omega}{\partial v} + \frac{\partial \Omega'}{\partial v'} \right) \sigma dS - \frac{1}{2\pi} \frac{\partial}{\partial X} \iint_{\tau_1} (\Omega - \Omega') \frac{\partial \sigma}{\partial v} dS \\ & - \frac{1}{2\pi} \frac{\partial}{\partial X} \iint_{\lambda} \left(\Omega \frac{\partial \sigma}{\partial v} - \sigma \frac{\partial \Omega}{\partial v} \right) dS - \frac{1}{2\pi} \frac{\partial}{\partial X} \iint_{\lambda'} \left(\Omega' \frac{\partial \sigma}{\partial v'} - \sigma \frac{\partial \Omega'}{\partial v'} \right) dS \end{aligned}$$

The integrations over τ are now in a form which may be interpreted directly in terms of known conditions over bodies with constant load or symmetrical section. The integration over λ and λ' can be handled by discussing the two cases shown in figure 2. If the leading edge is swept ahead of the Mach cone, as in figure 2(a), the value of Ω across the wedge springing from the leading edge is discontinuous, being zero immediately ahead of the wedge and a constant immediately behind it. It turns out, however, that in such cases the integration over λ just cancels the integration over λ' provided only that the constants are equal and opposite above and below the wing. If the leading edge is swept behind the leading edge as in figure 2(b) the value of u , v and w are all zero on the cone (i.e., continuous through the cone) and consequently, if Ω

represents one of these perturbation velocities, Ω , Ω' , $\frac{\partial \Omega}{\partial v}$, $\frac{\partial \Omega'}{\partial v'}$ are all zero on λ . Thus in either case there results the equation:

$$\Omega(X, Y, Z) = \frac{1}{2\pi} \frac{\partial}{\partial X} \iint_{\tau_1} \left(\frac{\partial \Omega}{\partial v} + \frac{\partial \Omega'}{\partial v'} \right) \sigma dS - \frac{1}{2\pi} \frac{\partial}{\partial X} \iint_{\tau_1} (\Omega - \Omega') \frac{\partial \sigma}{\partial v} dS \quad (14)$$

The counterpart of this equation for incompressible fluid flow is given on page 60 of reference 6.

Under the particular conditions for which

$$\frac{\partial \Omega}{\partial v} = - \frac{\partial \Omega'}{\partial v'} \quad (15)$$

over the surface τ equation (14) becomes

$$\Omega(X,Y,Z) = - \frac{1}{2\pi} \frac{\partial}{\partial X} \iint_{\tau_1} (\Omega - \Omega') \frac{\partial \sigma}{\partial v} dS \quad (16)$$

The restrictions imposed in equation (15) can be given physical significance after the functions Ω , Ω' and the surface τ have been given specific meanings. Consider first the case where τ is a lifting surface. Obviously the normal induced velocity w is a continuous function across τ . If Ω and Ω' are velocity potentials associated with the lifting surface,

$$w(X,Y,Z) = w'(X,Y,Z) = \frac{\partial \Omega}{\partial v} = - \frac{\partial \Omega'}{\partial v'}$$

and equation (15) is satisfied. If Ω denotes acceleration potential or perturbation velocity u , it is necessary to show that on the lifting surface

$$\frac{\partial u}{\partial v} = - \frac{\partial u'}{\partial v'}$$

This relation holds, however, for since $w(X,Y,Z) = w'(X,Y,Z)$ along τ , it follows that

$$\frac{\partial w}{\partial X} = \frac{\partial w'}{\partial X'}$$

and from the condition of irrotationality it is possible to express the gradient of w in the X -direction as the gradient of u normal to the surface, that is, in the directions of v and v' . Equation (16) is thus applicable directly to lifting surface theory in conjunction with either velocity or acceleration potentials. Application can also be made to the determination of pressure distribution over the surface of a symmetric airfoil at zero angle of attack. In this so-called nonlifting case the function Ω is set equal to the

induced velocity w , τ is the plane of symmetry of the airfoil, and equation (16) can be used to establish the boundary conditions provided equation (15) is satisfied. For this to be so $\partial w / \partial v$ must equal $-\partial w' / \partial v'$. But conditions of symmetry give $w(Z) = -w'(-Z)$ from which the equality is seen to hold.

Retransformation of Coordinates

Since

$$\sigma = \text{arc cosh} \frac{X - X_1}{\sqrt{(Y - Y_1)^2 + (Z - Z_1)^2}}$$

direct substitution into equation (16) gives

$$\Omega(X, Y, Z) = -\frac{1}{2\pi} \frac{\partial}{\partial X} \iint_{\tau_1} \frac{(\Omega - \Omega') (X - X_1) (Z - Z_1) dX_1 dY_1}{[(Y - Y_1)^2 + (Z - Z_1)^2] \sqrt{(X - X_1)^2 - (Y - Y_1)^2 - (Z - Z_1)^2}}$$

This solution applies to equation (5) and, in order to relate problems to the linearized equation (2), it is necessary to use the transformation of equations (3). If the point X_1, Y_1, Z_1 transforms to the point x_1, y_1, z_1 , it follows that

$$\Omega(x, y, z) = -\frac{1}{2\pi} \frac{\partial}{\partial x} \iint_{\tau_1} \frac{(\Omega - \Omega') (x - x_1) (z - z_1) dx_1 dy_1}{[(y - y_1)^2 + (z - z_1)^2] \sqrt{(x - x_1)^2 - \beta^2 [(y - y_1)^2 + (z - z_1)^2]}} \quad (17)$$

where

$$\beta^2 = M^2 - 1$$

APPLICATIONS

General Remarks

Applications in lifting surface theory may proceed along two possible lines depending upon the boundary conditions specified. In what is usually referred to as the direct problem the loading is given over the wing and the potential function of the flow field is calculated. From the potential function the shape of the aerodynamic surface supporting this load can be found relatively easily. The inverse problem concerns itself with the determination of the loading over a wing surface from a knowledge of the surface shape. In the following sections both of these problems will be considered. The direct problem will be discussed for various plan forms, the analysis proceeding directly from the expression for the potential function given in equation (17). The detailed discussion of the direct problem is justified by its application to the inverse problem where the loading over flat plates with rectangular, trapezoidal, and triangular plan forms is determined. The mathematics of the inverse problem is less straight forward since the analysis involves the introduction of elemental lifting surfaces with constant loading and the solution of an integral equation for each plan form. The cases considered are, however, of particular interest since the flat plate loading represents in thin airfoil theory the additional loading due to angle of attack which is associated with the particular plan form.

Uniformly Loaded Lifting Surfaces in Supersonic Flow

Infinite span wing.— In order to determine the induced velocities on the surface of an infinite span, uniformly loaded, supersonic lifting surface by means of the methods derived in the preceding section it is convenient to set Ω equal to the acceleration potential ϕ (reference 2). The lifting surface is, in this case, a surface of discontinuity for the function ϕ and corresponds to the surface τ_1 in equation (17). The discontinuity in the value of ϕ between the upper and lower surface is equal to

$$(\phi_u - \phi_l) = \frac{1}{\rho_0} (p_l - p_u)$$

where

ρ_0 density in the free stream

p_l static pressure on lower surface

p_u static pressure on upper surface

It follows that for the uniformly loaded wing in the plane $z_1=0$ the discontinuity in the acceleration potential is a constant, say C_0 . From equation (17)

$$\varphi(x,y,z) = -\frac{1}{2\pi} \frac{\partial}{\partial x} \iint \frac{C_0(x-x_1) z \, dx_1 \, dy_1}{[(y-y_1)^2+z^2] \sqrt{(x-x_1)^2-\beta^2[(y-y_1)^2+z^2]}} \quad (18)$$

A sketch of the airfoil plan form is given in figure 3 and two possible regions of integration are indicated. In all cases the integration with respect to y is performed between the limits at which the radical

$$\sqrt{(x-x_1)^2-\beta^2[(y-y_1)^2+z^2]}$$

vanishes while the integration with respect to x depends upon the manner in which the forecone of the point P intersects the discontinuity surface. Denoting the chord length of the airfoil by c , the following relations are obtained:

$$\begin{aligned} \varphi &= 0 && \text{when } x+\beta z < 0 \\ \varphi &= \pm \frac{1}{2} C_0 && \text{when } 0 \leq x+\beta z \leq c \\ \varphi &= 0 && \text{when } c < x+\beta z \end{aligned} \quad (19)$$

(When double signs are used, the upper sign refers always to the case where $z > 0$ and the lower sign corresponds to $z < 0$.)

The value of the acceleration potential is thus seen to be zero at all points in space except for those points lying within the region between the wedges extending back from the leading and trailing edges of the airfoil.

It is now possible to determine the induced velocities associated

with the acceleration potential just obtained. Since, in linear perturbation theory (reference 2)

$$\frac{\partial \phi}{\partial x} = V_0 \frac{\partial u}{\partial x}, \quad \frac{\partial \phi}{\partial y} = V_0 \frac{\partial v}{\partial x}, \quad \frac{\partial \phi}{\partial z} = V_0 \frac{\partial w}{\partial x} \quad (20)$$

where u, v, w are respectively the x, y, z components of the perturbation velocities, it follows that

$$u = \frac{1}{V_0} \phi$$

$$v = \frac{\partial}{\partial y} \int_{-\infty}^x \frac{1}{V_0} \phi(x_1, y, z) dx_1 \quad (21)$$

$$w = \frac{\partial}{\partial z} \int_{-\infty}^x \frac{1}{V_0} \phi(x_1, y, z) dx_1$$

The induced velocities for the infinite span airfoil result immediately from equations (19) and (21). If the upper sign of a double sign is again referred to the $z > 0$ case, the results may be written in the form

$$\left. \begin{aligned} u &= \pm \frac{C_0}{2V_0} \\ v &= 0 \\ w &= -\frac{C_0}{2V_0} \sqrt{M^2 - 1} \end{aligned} \right\} \text{for } 0 \leq x \mp \beta z < c \quad (22)$$

Since the vertical induced velocities are constant, it follows that the supersonic airfoil of infinite aspect ratio and uniform load distribution is a flat plate. The relations between this loading and angle of attack will be considered later.

Lifting surface with rectangular plan form.— The complete

discussion of the supersonic lifting surface with uniform loading and rectangular plan form is lengthened considerably by the fact that in calculating the acceleration potential at the point P with coordinates x, y, z it is necessary to distinguish between several regions in space in which the point may be located. These regions arise from consideration of the manner in which the fore-cone of the point P cuts the surface of discontinuity. The value of ϕ can be found with approximately equal facility in each of these regions but, since this paper is concerned primarily with effects on the surface of the airfoil, the solutions for pertinent regions only will be given here.

Figure 4 shows the rectangular plan form LL'T'T together with the coordinate system to be used. The dimensions of the wing are chosen so that the Mach cones extending back from the leading edge will not intersect within the boundaries of the wing. This restriction, which is not necessary but merely simplifies the analysis, implies that if b is the span of the wing and c the chord length, then

$$\tan \mu = \frac{1}{\sqrt{M^2 - 1}} < \frac{b}{2c} \quad (23)$$

where $\mu = \arcsin \frac{1}{M}$ is the so-called Mach angle of the stream and equal to the semivertex angles of the Mach cones.

The loading over the rectangular plan form is to be uniform so the expression $\phi_u - \phi_l$ is set equal to C_0 for $-\frac{1}{2}b \leq y_1 \leq \frac{1}{2}b$ and $0 \leq x_1 \leq c$. The acceleration potential, expressed as a function of x, y, z , is thus obtainable from equation (17) and the limits of integration must be determined from the position of P . From reasons of symmetry, only the portion of space from which $y \geq 0$ need be considered. Once the acceleration potential has been calculated, equations (21) may be used to calculate induced velocities. The results of such calculations are given and the same convention for double signs is used.

Region I_1 : Behind the leading-edge wedge, ahead of the trailing-edge wedge and bounded laterally by the inner sides of the two-tip Mach cones. The results in this region correspond to results obtained for the infinite span airfoil.

Thus

$$\begin{aligned}\phi &= \pm \frac{1}{2} C_0 \\ u &= \pm C_0 / 2V_0 \\ v &= 0 \\ w &= - \frac{C_0}{2V_0} \beta\end{aligned}\quad (24)$$

Region I_2 : Within the Mach cone coming from the leading-edge tip but outside the Mach cone from the trailing-edge tip. The integral of equation (17) must be broken into two parts. Denoting the integrand in the equation by the symbol I ,

$$\phi(x, y, z) = - \frac{1}{2\pi} \frac{\partial}{\partial x} \left[\int_0^{X_1} dx_1 \int_{Y_1}^{\frac{1}{2}b} I dy_1 + \int_{X_1}^{X_2} dx_1 \int_{Y_1}^{Y_2} I dy_1 \right] \quad (25)$$

where

$$\begin{aligned}Y_1 &= y - \frac{1}{\beta} \sqrt{(x-x_1)^2 - \beta^2 z^2} & X_1 &= x - \beta \sqrt{(y - \frac{1}{2}b)^2 + z^2} \\ Y_2 &= y + \frac{1}{\beta} \sqrt{(x-x_1)^2 - \beta^2 z^2} & X_2 &= x + \beta z\end{aligned}$$

After integration of equation (25) and application of equations (21)

$$\begin{aligned}\phi &= \frac{C_0}{2\pi} \left\{ \pm \frac{\pi}{2} + \arctan \frac{x (y - \frac{1}{2}b)}{z \sqrt{x^2 - \beta^2 [(y - \frac{1}{2}b)^2 + z^2]}} \right\} \\ u &= \frac{C_0}{2\pi V_0} \left\{ \pm \frac{\pi}{2} - \arctan \frac{x (y - \frac{1}{2}b)}{z \sqrt{x^2 - \beta^2 [(y - \frac{1}{2}b)^2 + z^2]}} \right\} \\ v &= \frac{C_0}{2\pi V_0} \frac{-z}{(y - \frac{1}{2}b)^2 + z^2} \sqrt{x^2 - \beta^2 [(y - \frac{1}{2}b)^2 + z^2]} \\ w &= \frac{C_0}{2\pi V_0} \left\{ - \frac{\beta\pi}{2} + \beta \arctan \frac{\beta (y - \frac{1}{2}b)}{\sqrt{x^2 - \beta^2 [(y - \frac{1}{2}b)^2 + z^2]}} \right. \\ &\quad \left. + \frac{(y - \frac{1}{2}b)}{(y - \frac{1}{2}b)^2 + z^2} \sqrt{x^2 - \beta^2 [(y - \frac{1}{2}b)^2 + z^2]} \right\}\end{aligned}\quad (26)$$

As a partial check on the expression for ϕ in equation (26) it can be seen that in the limit as $z \rightarrow 0$ the value of ϕ agrees with the result given in equation (24) on the wing while the value is zero off the wing.

The values of vertical induced velocity in the plane $z = 0$ are of particular interest since from a knowledge of the distribution of w the surface shape and local angle of attack corresponding to the imposed load distribution can be determined. The expressions for w for uniform loading will be particularly useful later when the load distribution is modified in order to obtain airfoils with specified induced velocities. Introducing the notation

$$\eta = \frac{(y - \frac{1}{2}b)\beta}{x} \quad (27)$$

the following results are obtained for the area covered by the tip cone:

(a) For $x < c$ and $-1 < \eta < +1$

$$w_{z=0} = \frac{C_o\beta}{2\pi V_o} \left\{ -\frac{\pi}{2} + \frac{\sqrt{1-\eta^2}}{\eta} + \arcsin \eta \right\} \quad (28)$$

For later reference it is desirable to have $w_{z=0}$ given in integral form. These expressions are

$$\begin{aligned} 0 < \eta < 1 \quad w_{z=0} &= \frac{-C_o\beta}{2\pi V_o} \int_{+1}^{\eta} \frac{\sqrt{1-\eta_1^2}}{\eta_1} d\eta_1 \\ -1 < \eta < 0 \quad w_{z=0} &= \frac{-C_o\beta}{2\pi V_o} \left(\pi + \int_{-1}^{\eta} \frac{\sqrt{1-\eta_1^2}}{\eta_1} d\eta_1 \right) \end{aligned} \quad (29)$$

Equations (28) and (29) indicate that the flow over the tip portion of the airfoil is of the type referred to as "conical flow." For this type of flow the values of induced downwash, aerodynamic loading, etc., are functions merely of the angle η . Busemann (reference 7), Stewart (reference 8) and Lagerstrom (reference 9) have developed analyses for certain plan forms which are postulated

on the existence of this type of solution and it appears that the theory can be shortened by such an approach. The chief value of the present method is most likely to be appreciated when no such assumptions can be made.

Tip of swept-forward lifting surface.— Consider the tip of a swept-forward supersonic lifting surface with uniform loading (fig. 5), the angle δ_0 between the leading edge and the X-axis and the angle δ_1 between the trailing edge and the X-axis both being less than the free-stream Mach angle μ . In carrying out an analysis it is necessary to distinguish between the type where the tip boundaries are behind the Mach cones and the type where the tip boundaries are ahead. The analyses of these two cases are of equivalent complexity, however, and can be handled with equal facility by the methods of this report. For all surfaces whose leading edges form an apex, only the case where the wing boundaries are behind the Mach cone will be considered. A Cartesian coordinate system is chosen as shown so that the origin lies at the vertex, the positive X-axis extending downstream, the Y-axis extending laterally, and the Z-axis being directed normal to the plane of the plan form and to the free-stream direction. The equations of the sides of the lifting surface are

$$y = 0$$

$$y = -x \tan \delta_0 = -\frac{\theta_0}{\beta} x$$

and

$$y = -(x-c) \tan \Delta_1 = -\frac{\theta_1}{\beta} (x-c)$$

The calculation of $\phi(x,y,z)$ again must be divided into several cases depending upon the location of the point $P(x,y,z)$ and again only solutions pertinent to the conditions on the lifting surface itself will be given. In the results listed below are included the general form of the integral for the acceleration potential $\phi(x,y,z)$ and the explicit expressions for $\phi(x,y,z)$. The induced velocities, however, are given only in the plane $z=0$, as the integration to obtain a general expression is difficult. The velocities in the $z=0$ plane, which are sufficient for the purpose of this investigation, can be obtained from a simpler integration since, for the integral involved,

$$\lim_{z \rightarrow 0} \int I(x,y,z) dx = \int I(x,y,0) dx$$

This simplification was used in the analysis of most of the lifting surfaces investigated. As before, it is assumed that the discontinuity

in ϕ is equal to C_0 . Moreover, the expressions for $w_{z=0}$ are given in terms of the variables η and ω where

$$\eta = \frac{\beta y}{x} \text{ and } \omega = 1 - \frac{c}{x} \quad (30)$$

In this manner the solution is shown to be conical in the region ahead of the trailing-tip Mach cone (fig. 5). For points behind this Mach cone the flow is not conical but a function of both η and ω .

Region I₁: Inside the leading-tip Mach cone and ahead of the plane $y + (\theta_0/\beta)x = 0$ with $y < 0$. From equation (17)

$$\phi(x, y, z) = -\frac{1}{2\pi} \frac{\partial}{\partial x} \left[\int_0^{X_2} dx_1 \int_{Y_1}^0 Idy_1 + \int_{X_2}^{X_1} dx_1 \int_{Y_1}^{Y_2} Idy_1 \right] \quad (31)$$

where

$$Y_1 = -\frac{\theta_0}{\beta} x$$

$$Y_2 = y - \frac{1}{\beta} \sqrt{(x-x_1)^2 - \beta^2 z^2}$$

$$X_1 = \frac{(x + \beta \theta_0 y) - \beta \sqrt{(y + \frac{\theta_0}{\beta} x)^2 + z^2 (1 - \theta_0^2)}}{1 - \theta_0^2}$$

$$X_2 = x - \beta \sqrt{y^2 + z^2}$$

After integration

$$\phi = \frac{C_0}{2\pi} \left[-\arctan \frac{xy}{z \sqrt{x^2 - \beta^2 (y^2 + z^2)}} + \arctan \frac{xy + x^2 \frac{\theta_0}{\beta} - \beta z^2 \theta_0}{z \sqrt{x^2 - \beta^2 (y^2 + z^2)}} \right] \quad (32)$$

In this region, $-1 < \eta < -\theta_0$,

$$w_{z=0} = -\frac{C_0 \beta \theta_0}{2\pi V_0} \int_{-1}^{\eta} \frac{\sqrt{1 - \eta_1^2}}{\eta_1^2 (\eta_1 + \theta_0)} d\eta_1 \quad (33)$$

$$= \frac{C_0 \beta}{2\pi V_0} \left[\frac{\sqrt{1 - \eta^2}}{\eta} - \frac{1}{\theta_0} \operatorname{arc} \cosh \frac{-1}{\eta} + \frac{\sqrt{1 - \theta_0^2}}{\theta_0} \operatorname{arc} \cosh \frac{-(1 + \theta_0 \eta)}{(\theta_0 + \eta)} \right]$$

Region I₂: Inside the leading-tip Mach cone, outside the trailing-tip Mach cone and between the planes $y = 0$ and $y + (\theta_0/\beta)x = 0$. Using the notation of equation (32) for X_1 , Y_1 , etc., the acceleration potential is

$$\phi(x, y, z) = -\frac{1}{2\pi} \frac{\partial}{\partial x} \left(\int_0^{X_1} dx_1 \int_{Y_1}^0 Idy_1 + \int_{X_1}^{X_2} dx_1 \int_{Y_2}^0 Idy_1 + \int_{X_2}^{X_3} dx_1 \int_{Y_2}^{Y_3} Idy_1 \right) \quad (34)$$

where

$$Y_3 = y + \frac{1}{\beta} \sqrt{(x-x_1)^2 - \beta^2 z^2}$$

$$X_3 = x + \beta z$$

Integration yields

$$\phi = \frac{C_0}{2\pi} \left[-\arctan \frac{xy}{z \sqrt{x^2 - \beta^2 (y^2 + z^2)}} + \arctan \frac{xy + \frac{x^2 \theta_0}{\beta} - \beta z^2 \theta_0}{z \sqrt{x^2 - \beta^2 (y^2 + z^2)}} \right] \quad (35)$$

For the region considered $-\theta_0 < \eta < 0$ and in this case

$$w_{z=0} = -\frac{C_0 \beta \theta_0}{2\pi V_0} \int_{-1}^{\eta} \frac{\sqrt{1-\eta_1^2}}{\eta_1^2 (\eta_1 + \theta_0)} d\eta_1 \quad (36)$$

$$= \frac{C_0 \beta}{2\pi V_0} \left[\frac{\sqrt{1-\eta^2}}{\eta} - \frac{1}{\theta_0} \operatorname{arccosh} \frac{-1}{\eta} + \frac{\sqrt{1-\theta_0^2}}{\theta_0} \operatorname{arccosh} \frac{(1+\theta_0 \eta)}{(\theta_0 + \eta)} \right]$$

Region I₃: Inside the leading-tip Mach cone, outside the trailing-tip Mach cone and for values of y greater than zero. It follows that

$$\phi(x, y, z) = -\frac{1}{2\pi} \frac{\partial}{\partial x} \left[\int_0^{X_1} dx_1 \int_{Y_1}^0 Idy_1 + \int_{X_1}^{X_2} dx_1 \int_{Y_2}^0 Idy_1 \right] \quad (37)$$

The integration yields

$$\varphi = \frac{C_0}{2\pi} \left[-\arctan \frac{xy}{z \sqrt{x^2 - \beta^2 (y^2 + z^2)}} + \arctan \frac{xy + \frac{x^2 \theta_0}{\beta} - \beta z^2 \theta_0}{z \sqrt{x^2 - \beta^2 (y^2 + z^2)}} \right] \quad (38)$$

In this region, $0 < \eta < 1$

$$\begin{aligned} w_{z=0} &= \frac{-C_0 \beta \theta_0}{2\pi V_0} \int_1^\eta \frac{\sqrt{1-\eta_1^2}}{\eta_1^2 (\eta_1 + \theta_0)} d\eta_1 \\ &= \frac{C_0 \beta}{2\pi V_0} \left[\frac{\sqrt{1-\eta^2}}{\eta} - \frac{1}{\theta_0} \operatorname{arcosh} \frac{1}{\eta} + \frac{\sqrt{1-\theta_0^2}}{\theta_0} \operatorname{arcosh} \frac{(1+\theta_0\eta)}{(\theta_0+\eta)} \right] \quad (39) \end{aligned}$$

Region I_4 : Inside both tip Mach cones and ahead of the plane $y = -(\theta_1/\beta)(x-c)$ for $y < 0$. The solution in this region is greatly simplified by use of the fact that for linear differential equations any algebraic sum of solutions will be another solution of the equation. Since the differential equation for the acceleration potential is linear, this property can be applied to obtain a solution for this section of the airfoil by subtracting from the expressions given for region I_2 the expressions given for region I_1 except that in the latter region the variables are changed such that δ_1 replaces δ_0 and $x-c$ replaces x . From this operation it follows:

$$\begin{aligned} \varphi &= \frac{C_0}{2\pi} \left[-\arctan \frac{xy}{z \sqrt{x^2 - \beta^2 (y^2 + z^2)}} + \arctan \frac{xy + \frac{\theta_0}{\beta} x^2 - \beta \theta_0 z^2}{z \sqrt{x^2 - \beta^2 (y^2 + z^2)}} \right. \\ &\quad \left. + \arctan \frac{(x-c)y}{z \sqrt{(x-c)^2 - \beta^2 (y^2 + z^2)}} - \arctan \frac{(x-c)y + \frac{\theta_1}{\beta} (x-c)^2 - \beta \theta_1 z^2}{z \sqrt{(x-c)^2 - \beta^2 (y^2 + z^2)}} \right] \quad (40) \\ w_{z=0} &= \frac{C_0 \beta}{2\pi V_0} \left[\frac{\sqrt{1-\eta^2}}{\eta} - \frac{1}{\theta_0} \operatorname{arcosh} \frac{1}{\eta} + \frac{\sqrt{1-\theta_0^2}}{\theta_0} \operatorname{arcosh} \frac{(1+\theta_0\eta)}{(\theta_0+\eta)} \right. \\ &\quad \left. - \frac{\sqrt{\omega^2 - \eta^2}}{\eta} + \frac{1}{\theta_1} \operatorname{arcosh} \left(-\frac{\omega}{\eta} \right) - \frac{\sqrt{1-\theta_1^2}}{\theta_1} \operatorname{arcosh} \left(-\frac{\omega + \theta_1 \eta}{\theta_1 \eta + 1} \right) \right] \quad (41) \end{aligned}$$

Lifting surface with trapezoidal plan form.— The linear property of the differential equation may be used to advantage in determining the flow about a trapezoidal lifting surface with uniform lift distribution, since the boundary conditions within the plan form of the airfoil are obviously satisfied when the acceleration potential for a triangular tip is subtracted from the potential for the rectangular surface.

Suppose (fig. 6) the angle of rake of the trapezoid is δ_0 and that δ_0 is less than the Mach angle μ . The acceleration potential will be identical, over the central portion of the surface, to that for the lifting surface of infinite aspect ratio. Over the parts of the surface which are blanketed by the tip Mach cones the flow will, however, be modified. Because of symmetry the determination of this modification need only be carried out on one side of the figure.

If the coordinate axes are chosen as shown in figure 6, the lateral boundary of the lifting surface is

$$y = -x \tan \delta_0 = -\frac{\theta_0 x}{\beta}$$

It has been shown that both the rectangular plan form and the triangular plan form experience conical type flow over the region within the tip Mach cones. Thus, the variable η defined in equation (30) may be used.

Region I_1 : Inside the Mach cone originating at the leading-edge tip and ahead of the plane $y = -\frac{\theta_0}{\beta} x$ (i.e., on the surface of the wing), $-1 < \eta < -\theta_0$:

$$w_{z=0} = \frac{C_0 \beta}{2\pi V_0} \left(-\pi - \int_{-1}^{\eta} \frac{\sqrt{1-\eta_1^2}}{\eta_1} \frac{d\eta_1}{\eta_1 + \theta_0} \right) \quad (42)$$

$$= \frac{C_0 \beta}{2\pi V_0} \left[-\frac{\pi}{2} + \arctan \frac{\eta}{\sqrt{1-\eta^2}} + \frac{1}{\theta_0} \operatorname{arc cosh} \left(-\frac{1}{\eta} \right) - \frac{\sqrt{1-\theta_0^2}}{\theta_0} \operatorname{arc cosh} \frac{-(1+\theta_0 \eta)}{(\theta_0 + \eta)} \right]$$

Region I_2 : Inside the Mach cone originating at the leading-edge tip, outside the Mach cone originating at the trailing-edge tip, and between the planes $y = 0$ and $y = -\frac{\theta_0}{\beta} x$, $-\theta_0 < \eta < 0$:

$$w_{z=0} = \frac{C_0 \beta}{2\pi V_0} \left(-\pi - \int_{-1}^{\eta} \frac{\sqrt{1-\eta_1^2}}{\eta_1} \frac{d\eta_1}{\eta_1 + \theta_0} \right) \quad (43)$$

$$= \frac{C_0 \beta}{2\pi V_0} \left[-\frac{\pi}{2} + \arctan \frac{\eta}{\sqrt{1-\eta^2}} + \frac{1}{\theta_0} \operatorname{arcosh} \left(-\frac{1}{\eta} \right) - \frac{\sqrt{1-\theta_0^2}}{\theta_0} \operatorname{arcosh} \frac{1+\theta_0 \eta}{\theta_0 + \eta} \right]$$

Swept-back lifting surface.— As another example of the way in which the linearity of the differential equation may be utilized to obtain further solutions, the induced vertical velocities for a swept-back wing will be determined for the case in which the leading and trailing edges lie behind their respective Mach cones (fig. 7). The boundaries of the plan form shall be given by the equations

$$y = -\frac{\theta_0}{\beta} x, \quad y = \frac{\theta_0}{\beta} x$$

$$y = -\frac{\theta_1}{\beta} (x-c), \quad y = \frac{\theta_1}{\beta} (x-c)$$

The flow will be conical ahead of the trailing-edge Mach cone where the induced velocities can be expressed in terms of the variable η . Behind the trailing-edge Mach cone the flow will not be conical but will be expressible in terms of the variables η and $\omega = 1 - \frac{c}{x}$.

Consider first the region of conical flow. In order to determine $w_{z=0}$ for a given value of η it is possible to consider separately the induced effects reduced by each half of the surface. But in this region ahead of the trailing-edge Mach cone, the induced velocities arising from one half of the surface are given by the formula for a similar region on the swept-forward surface. For reasons of symmetry the results for the entire swept-back lifting

surface need only be given for values of η within the limits $-1 < \eta < 0$.

In the region where the flow is not conical the solution will be built up of a combination of solutions obtained from the regions of conical flow.

Region I_1 : Inside the leading-edge Mach cone, outside the trailing-edge Mach cone and ahead of the plane $y = -\frac{\theta_0}{\beta} x$. From equation (39) the value of $w_{z=0}$ resulting from the induced effects of the left-hand portion of the triangle can be determined and from equation (33) the induced effect produced by the right-hand part of the triangle is given provided the sign of η is changed. Combining these two results, it follows that

$$\begin{aligned} w_{z=0} &= \frac{C_0 \beta}{2\pi V_0} \left[-\frac{2}{\theta_0} \operatorname{arc} \cosh \frac{-1}{\eta} + \frac{\sqrt{1-\theta_0^2}}{\theta_0} \left(\operatorname{arc} \cosh \frac{1+\theta_0 \eta}{\theta_0 + \eta} + \operatorname{arc} \cosh \frac{1-\theta_0 \eta}{\theta_0 - \eta} \right) \right] \\ &= \frac{C_0 \beta}{2\pi V_0} \left(-\frac{2}{\theta_0} \operatorname{arc} \cosh \frac{-1}{\eta} + \frac{\sqrt{1-\theta_0^2}}{\theta_0} \operatorname{arc} \cosh \frac{2-\theta_0^2-\eta^2}{\theta_0^2-\eta^2} \right) \quad (44) \end{aligned}$$

Region I_2 : Inside the leading-edge Mach cone, outside the trailing-edge Mach cone, and between the planes $y = -\frac{\theta_0}{\beta} x$ and $y = 0$. In the same manner as in the preceding case equations (39) and (36) can be combined to give this solution. Thus:

$$\begin{aligned} w_{z=0} &= \frac{C_0 \beta}{2\pi V_0} \left\{ -\frac{2}{\theta_0} \operatorname{arc} \cosh \frac{-1}{\eta} + \frac{\sqrt{1-\theta_0^2}}{\theta_0} \left[\operatorname{arc} \cosh \frac{1-\theta_0 \eta}{\theta_0 - \eta} + \operatorname{arc} \cosh -\left(\frac{1+\theta_0 \eta}{\theta_0 + \eta} \right) \right] \right\} \\ &= \frac{C_0 \beta}{2\pi V_0} \left(-\frac{2}{\theta_0} \operatorname{arc} \cosh \frac{-1}{\eta} + \frac{\sqrt{1-\theta_0^2}}{\theta_0} \operatorname{arc} \cosh \frac{2-\theta_0^2-\eta^2}{\eta^2-\theta_0^2} \right) \quad (45) \end{aligned}$$

The integral expression for $w_{z=0}$ in this conical region, both on and off the surface, reduces to the same equation which is

$$w_{z=0} = \frac{\beta C_0 \theta_0}{2\pi V_0} \int_{-1}^{\eta} \frac{\sqrt{1-\eta_1^2}}{\eta_1} \frac{2}{\theta_0^2 - \eta_1^2} d\eta_1 \quad (46)$$

Region I_3 : Inside both Mach cones and ahead of the plane

$y = -\frac{\theta_1}{\beta} (x-c)$. The solution in this region can be produced by subtracting from the value of $w_{z=0}$ given for region I_2 the value of $w_{z=0}$ given for region I_1 except that in the latter case δ_0 is replaced by δ_1 and x by $(x-c)$. Thus, for this region,

$$w_{z=0} = \frac{C_0 \beta}{2\pi V_0} \left[\frac{-2}{\theta_0} \operatorname{arc} \cosh \frac{-1}{\eta} + \frac{\sqrt{1-\theta_0^2}}{\theta_0} \operatorname{arc} \cosh \frac{2-\theta_0^2-\eta^2}{\eta^2-\theta_0^2} \right. \\ \left. + \frac{2}{\theta_1} \operatorname{arc} \cosh -\frac{\omega}{\eta} - \frac{\sqrt{1-\theta_1^2}}{\theta_1} \operatorname{arc} \cosh \frac{(2-\theta_0^2)\omega^2-\eta^2}{\theta_0^2\omega^2-\eta^2} \right] \quad (47)$$

Although the uniformly loaded lifting surface was the only prescribed loading analyzed, it should be noted that the basic integration leading to a solution of this type of problem (equation (17)) is in no way restricted to a uniform load. Arbitrary loadings that may or may not be analytic functions of x and y can be specified and the problem therefore becomes one of technique in integration. The solutions for the uniformly loaded surfaces, however, are particularly useful. By methods of superposition these solutions can be used to obtain the surface loading for specified plan forms (the inverse problem) as will be illustrated in the following section.

Load Distributions on Flat-Plate Lifting Surfaces in Supersonic Flow

Infinite span wing.— Since the vertical induced velocity is constant for the supersonic airfoil of infinite aspect ratio (equation (22)) and uniform load it follows that the airfoil is a flat plate. This property distinguishes the infinite aspect ratio problem from all other plan forms considered, for the load distribution must be modified in the latter cases so that twist and camber are removed from the wing to obtain a flat plate.

Denoting the angle of attack of the airfoil by α ,

$$\alpha = -\frac{w}{V_0} = \frac{C_0}{2V_0^2} \sqrt{M^2-1} \quad (48)$$

Moreover,

$$p_l - p_u = \rho_o (\phi_u - \phi_l) = \rho_o C_o$$

and, setting

$$\frac{p_l - p_u}{\frac{1}{2} \rho_o V_o^2} = \frac{\Delta p}{q}$$

it follows that

$$\frac{\Delta p}{q} = \frac{2C_o}{V_o^2} \quad (49)$$

Eliminating C_o between equations (48) and (49)

$$\frac{\Delta p}{q} = \frac{4\alpha}{\sqrt{M^2 - 1}} \quad (50)$$

The result given in equation (50) is the well-known Ackeret expression developed in reference 10. The derivation here follows the approach of Prandtl (reference 2).

Rectangular plan form.— Since the vertical induced velocity for the uniformly loaded supersonic airfoil of rectangular plan form is not constant over the portion of the wing covered by the tip Mach cones, it is necessary to modify the load distribution within this region in order to get a flat plate. The determination of the required load distribution will be shown to depend on the solution of an integral equation and subsequent problems dealing with other plan forms will, from a mathematical standpoint, be similar in form.

The rectangular plan form will be thought of as being built of superimposed trapezoidal lifting surfaces with variable angles of rake (fig. 8), each trapezoidal surface having a constant load distribution but with loading allowed to vary with the variable rake angle δ .

Since the flow over the part of the airfoil within the Mach cone is conical, it is possible to express $w_{z=0}$ as a function of η where

$$\eta = \frac{\beta y}{x}$$

Setting

$$\beta \tan \delta = \theta$$

and using equations (42) and (43)

$$\frac{2\pi V_0 w(\eta)_{z=0}}{\beta} = \int_{\theta=0}^{\theta=1} C'(\theta) \left(-\pi - \int_{-1}^{\eta} \frac{\sqrt{1-\eta_1^2}}{\eta_1} \frac{d\eta_1}{\eta_1+\theta} \right) d\theta \quad (51)$$

where $C'(\theta) = \phi_u - \phi_l$ for the single trapezoidal surface with rake angle δ .

The solution of the problem depends on the determination of a function $C'(\theta)$ which, when substituted in equation (51), will yield a constant value of $w_{z=0}(\eta)$; that is, a value of $w_{z=0}$ independent of the variable η . Imposing the condition that

$$\frac{dw_{z=0}}{d\eta} = 0$$

the problem resolves itself into one of solving the equation

$$0 = \frac{d}{d\eta} \int_{\theta=0}^{\theta=1} C'(\theta) d\theta \int_{-1}^{\eta} \frac{\sqrt{1-\eta_1^2}}{\eta_1} \frac{d\eta_1}{\eta_1+\theta}$$

Introducing the notation

$$\int_{-1}^{\eta} \frac{\sqrt{1-\eta_1^2}}{\eta_1} \frac{d\eta_1}{\eta_1+\theta} = G_1(\eta, \theta)$$

the integral equation may be written in the form

$$0 = \lim_{\epsilon \rightarrow 0} \left[\frac{d}{d\eta} \int_{\theta=0}^{\theta=-\eta-\epsilon} C'(\theta) G_1(\eta, \theta) d\theta + \frac{d}{d\eta} \int_{\theta=-\eta+\epsilon}^{\theta=1} C'(\theta) G_1(\eta, \theta) d\theta \right]$$

where the singularity in the integrand necessitates the use of the infinitesimal ϵ . Taking the derivative,

$$0 = \lim_{\epsilon \rightarrow 0} \left[\int_0^{-\eta-\epsilon} C'(\theta) \frac{\partial G_1}{\partial \eta} d\theta + \int_{-\eta+\epsilon}^1 C'(\theta) \frac{\partial G_1}{\partial \eta} d\theta - C'(-\eta-\epsilon) G_1(\eta, -\eta-\epsilon) + C'(-\eta+\epsilon) G_1(\eta, -\eta+\epsilon) \right]$$

It is obvious from equations (42) and (43) that, if $C'(\theta)$ is a continuous function,

$$\lim_{\epsilon \rightarrow 0} \left[-C'(-\eta-\epsilon) G_1(\eta, -\eta-\epsilon) + C'(-\eta+\epsilon) G_1(\eta, -\eta+\epsilon) \right] = 0$$

Hence

$$0 = \frac{\sqrt{1-\eta_1^2}}{\eta_1} \int_0^1 \frac{C'(\theta) d\theta}{\eta_1 + \theta} \quad (52)$$

and the solution can be written in the form

$$C'(\theta) = \frac{C_1}{\sqrt{\theta(1-\theta)}} \quad (53)$$

where C_1 is a constant to be determined later. Substituting from equation (53) into equation (51)

$$\frac{2\pi V_{ow}(\eta)_{z=0}}{\beta C_1} = - \int_0^1 \frac{\pi d\theta}{\sqrt{\theta(1-\theta)}} - \int_0^1 \frac{d\theta}{\sqrt{\theta(1-\theta)}} \int_{-1}^{\eta} \frac{\sqrt{1-\eta_1^2}}{\eta_1(\eta_1+\theta)} d\eta_1 \quad (54)$$

The region of integration in the η_1, θ plane for the double integral of equation (54) is shown as the cross-hatched area of figure 9, a singularity in the integrand occurring along the line $\theta = -\eta_1$. Rewriting the equation and reversing the order of integration in the double integral,

$$\begin{aligned} \frac{2\pi V_{ow}(\eta)_{z=0}}{\beta C_1} = & -2\pi \arcsin \sqrt{\theta} \Big|_0^1 - \lim_{\epsilon \rightarrow 0} \int_{-1}^{\eta} \frac{\sqrt{1-\eta_1^2}}{\eta_1} d\eta_1 \left[\int_0^{-\eta_1-\epsilon} \frac{d\theta}{(\eta_1+\theta)\sqrt{\theta(1-\theta)}} \right. \\ & \left. + \int_{-\eta_1+\epsilon}^1 \frac{d\theta}{(\eta_1+\theta)\sqrt{\theta(1-\theta)}} \right] \end{aligned}$$

By means of the substitution

$$\theta = z^2, \eta_1 = -p^2$$

the bracketed expression in this equation can be shown to vanish for all values of η_1 between zero and -1 so that, finally,

$$w_{z=0} = \frac{-C_1\beta}{2\pi V_0} \left[2\pi \arcsin \sqrt{\theta} \right]_0^1 = \frac{-C_1\beta\pi}{2V_0} \quad (55)$$

Since the trapezoidal lifting surfaces are superimposed, the loading $C(\theta)$ over the resultant rectangular plan form satisfies the relation

$$\frac{dC(\theta)}{d\theta} = C'(\theta) = \frac{C_1}{\sqrt{\theta(1-\theta)}}$$

Imposing the condition that $C(\theta) = 0$ at $\theta = 0$, it follows that

$$C(\theta) = 2 C_1 \arcsin \sqrt{\theta} \quad (56)$$

This equation gives the incremental change of acceleration potential between the upper and lower lifting surface of the rectangular wing. As a result the increment in pressure is

$$p_l - p_u = \rho_0 (\phi_u - \phi_l) = 2\rho_0 C_1 \arcsin \sqrt{\theta}$$

Expressing the pressure difference in nondimensional terms,

$$\frac{p_l - p_u}{\frac{1}{2}\rho_0 V_0^2} = \frac{\Delta p}{q} = \frac{4C_1}{V_0^2} \arcsin \sqrt{\theta} \quad (57)$$

The constant C_1 may be eliminated between equations (55) and (57) and as a consequence

$$\frac{\Delta p}{q} = - \frac{w_{z=0}}{V_0} \frac{8}{\beta\pi} \arcsin \sqrt{\theta}$$

Since the angle of attack α of the airfoil is by definition equal to $-\frac{w_{z=0}}{V_0}$, the final expression for the loading, in coefficient form, over the outer portions of the rectangular wing is

$$\frac{\Delta p}{q} = \frac{8\alpha}{\pi\sqrt{M^2-1}} \arcsin \sqrt{\theta} \quad (58)$$

The general approach used to obtain this result is similar to that used by Schlichting (reference 11). The error in Schlichting's final result has been noted by Busemann and others.

Lift coefficient C_L for an arbitrary wing is defined by the relation

$$C_L = \frac{L}{S_0 q} = \frac{1}{S_0} \iint \frac{\Delta p}{q} dS \quad (59)$$

where

L total lift of the wing

dS element of area on the wing

S_0 total area of wing

For the rectangular wing the values of $\Delta p/q$ over the tip and center sections are given by equations (58) and (50). As a result of this integration

$$C_L = \frac{4\alpha}{\beta} \left(1 - \frac{1}{2\beta A}\right) \quad (60)$$

where A is the aspect ratio and by definition equal to the ratio of the square of the span and the wing area. As a final conclusion the lift curve slope of the wing is

$$\frac{dC_L}{d\alpha} = \frac{4}{\beta} \left(1 - \frac{1}{2\beta A}\right) \quad (61)$$

Trapezoidal plan form.— The results given in equations (58) and (61) are capable of generalization to the case of the flat plate having trapezoidal plan form and with rake angle δ_0 less than the

Mach angle of the stream. For such a configuration the airfoil is again blanketed in part by the tip Mach cones and the loading in this outer section of the airfoil must be adjusted properly to give constant induced vertical velocity. Superposition of trapezoidal lifting surfaces with loadings varying with rake angle δ can again be used and the conical nature of the flow employed. Setting

$$\eta = \beta y/x$$

$$\theta = \beta \tan \delta$$

$$\theta_o = \beta \tan \delta_o$$

equations (42) and (43) give the expression

$$\frac{2\pi V_o w(\eta)_{z=0}}{\beta} = \int_{\theta=\theta_o}^{\theta=1} C'(\theta) \left(-\pi \int_{-1}^{\eta} \frac{\sqrt{1-\eta_1^2}}{\eta_1} \frac{d\eta_1}{\eta_1 + \theta} \right) d\theta \quad (62)$$

where $C'(\theta) = \varphi_u - \varphi_l$ for the single trapezoidal surface with rake angle δ .

The analysis in this case follows along lines directly analogous to that used for the rectangular surface. For the present configuration the loading function for the superimposed trapezoids is given by the relation

$$C'(\theta) = \frac{C_1}{\sqrt{(\theta - \theta_o)(1 - \theta)}} \quad (63)$$

and by means of the substitution

$$z^2 = \frac{\theta - \theta_o}{1 - \theta_o}, \quad -p^2 = \frac{\eta_1 + \theta_o}{1 - \theta_o}$$

the integration to obtain $w_{z=0}$ can be simplified to give, as a final result,

$$w_{z=0} = -\frac{\beta C_1}{2\pi V_o} \int_{\theta_o}^1 \frac{\pi d\theta}{\sqrt{(\theta - \theta_o)(1 - \theta)}} = -\frac{\beta C_1 \pi}{2V_o} \quad (64)$$

The loading $C(\theta)$ over the resultant trapezoidal plan form can be found from the relation

$$\frac{dC(\theta)}{d\theta} = \frac{C_1}{\sqrt{(\theta - \theta_0)(1 - \theta)}}$$

From the boundary condition that $C(\theta) = 0$ at $\theta = \theta_0$ it follows that

$$C(\theta) = 2C_1 \arcsin \sqrt{\frac{\theta - \theta_0}{1 - \theta_0}}$$

and

$$\frac{p_l - p_u}{\frac{1}{2}\rho_0 V_0^2} = \frac{\Delta p}{q} = \frac{4C_1}{V_0^2} \arcsin \sqrt{\frac{\theta - \theta_0}{1 - \theta_0}} \quad (65)$$

Elimination of C_1 between equations (64) and (65) and introduction of angle of attack α for $-w_{z=0}/V_0$ gives as aerodynamic loading over the portion of the airfoil within the tip Mach cones the expression

$$\frac{\Delta p}{q} = \frac{8\alpha}{\pi\beta} \arcsin \sqrt{\frac{\theta - \theta_0}{1 - \theta_0}}, \quad 0 \leq \theta_0 \leq 1 \quad (66)$$

Figure 10 indicates the variation of the loading over the tip section of the trapezoid. The variable $(\beta/\alpha)(\Delta p/q)$ is plotted against $\beta \tan \delta$ for $\beta \tan \delta_0$ equal to 0, 0.3, and 0.6. The curve for $\beta \tan \delta_0 = 0$ corresponds to the case of the rectangular wing and shows results in agreement with equation (58).

By means of equation (59) together with equations (66) and (50) the lift coefficient of the trapezoidal wing is expressible in the form

$$C_L = \frac{4\alpha}{\beta} \left(\frac{1 - \frac{c}{2b} \tan \delta_0 - \frac{c}{2b} \tan \mu}{1 - \frac{c}{b} \tan \delta_0} \right) \quad (67)$$

Introducing the aspect ratio A of the wing where

$$A = \frac{b}{c(1 - \frac{c}{b} \tan \delta_o)}$$

lift coefficient can also be written

$$C_L = \frac{2\alpha}{\beta} \left[1 + \frac{Ac}{b} \left(1 - \frac{c}{\beta b} \right) \right] \quad (68)$$

From equation (67),

$$\frac{dC_L}{d\alpha} = \frac{4}{\beta} \left(\frac{1 - \frac{c}{2b} \tan \delta_o - \frac{c}{2b} \tan \mu}{1 - \frac{c}{b} \tan \delta_o} \right) \quad (69)$$

In figure 11, $\beta \frac{dC_L}{d\alpha}$ is plotted as a function of $A\beta$ for $\theta_o = 0, \frac{1}{2},$ and 1. The curve for $\theta_o = 0$ agrees with results given by equation (61) for the rectangular wing. All curves are terminated at values of $A\beta$ for which the tip Mach cones intersect on the trailing edge of the wing.

Triangular plan form, type 1.— The pressure distribution over triangular lifting surfaces with constant induced vertical velocities will be developed in the following three sections. These plan forms are indicated in figures 12(a), 12(b), and 12(c) and shall be denoted respectively as types 1, 2, and 3. Types 1 and 2 are actually special cases of type 3; namely, the cases where one leading edge is parallel to the free stream, and where both leading edges make equal angles with the stream direction. Type 3 includes any plan form which has leading edges swept behind the Mach cone but on opposite sides of an axis drawn through the vertex of the triangle and parallel to the free stream; and, further, has a trailing edge such that the Mach cones from either tip do not cross the surface of the wing. The principal reason for considering the three types separately is to show the manner in which the spanwise loading appears in the solution of the problem. In types 1 and 2 the proper load distribution is found readily while the final type requires a more careful treatment.

In order to determine the load distribution over the airfoil it will be convenient to use a differential element over which the

loading is constant. The elements may then be summed and the distribution of loading adjusted so that the induced vertical velocity at any point on the total lifting surface is constant. For the triangular plan forms it is possible to assume that conical flow exists and the analysis may be carried out using the angular coordinates which have already been introduced.

Figure 13 shows the elemental lifting surface to be used. The sides of the element extend back from the tip of the Mach cone; making angles δ and $\delta + \Delta\delta$ with the positive X-axis or free-stream direction. Corresponding to previous notation, the relations $\theta = \beta \tan \delta$ and $\theta + \Delta\theta = \beta \tan (\delta + \Delta\delta)$ are used. The vertical velocity induced by the element of surface may be denoted by Δw and it follows that

$$\Delta w = w(\theta + \Delta\theta, \eta) - w(\theta, \eta)$$

where $w(\theta, \eta)$ and $w(\theta + \Delta\theta, \eta)$ are the velocities induced by the triangular-tip surfaces with constant loading and with tip angles equal to δ and $\delta + \Delta\delta$, respectively. Applying a limiting process,

$$\lim_{\Delta\theta \rightarrow 0} \frac{\Delta w}{\Delta\theta} = \lim_{\Delta\theta \rightarrow 0} \frac{w(\theta + \Delta\theta, \eta) - w(\theta, \eta)}{\Delta\theta} = \frac{\partial w}{\partial \theta} \quad (70)$$

It follows that $w_{z=0}$ for the resultant lifting surface will be evaluated by an integration with respect to θ . If $\partial w_{z=0} / \partial \theta$ can be expressed in the form of an integral with respect to η_1 , the relation for $w_{z=0}$ will then be similar to those given in equations (51) and (62) for the previous plan forms and the expectation will be that the function $C(\theta)$ can be determined to give constant induced vertical velocity.

The method of attack just outlined is postulated on the existence of an integral expression for $\partial w_{z=0} / \partial \theta$. Such an expression is obtainable directly from the integrals in equations (33), (36), and (39); however, the alternate relations given in these equations lead more directly to the proper form. Proceeding to this calculation, the following results are obtained for the elemental lifting surface;

For $-\theta < \eta < 0$

$$\frac{\partial w_{z=0}}{\partial \theta} = \frac{\beta C_1}{2\pi V_0} \left[\frac{1}{\theta^2} \operatorname{arc} \cosh \frac{-1}{\eta} - \frac{1}{\theta^2 \sqrt{1-\theta^2}} \operatorname{arc} \cosh \frac{1+\theta\eta}{\theta+\eta} - \frac{\sqrt{1-\eta^2}}{\theta(\theta+\eta)} \right] \quad (71)$$

for $-1 < \eta < -\theta$

$$\frac{\partial w_{z=0}}{\partial \theta} = \frac{\beta C_1}{2\pi V_0} \left[\frac{1}{\theta^2} \operatorname{arc} \cosh \frac{-1}{\eta} - \frac{1}{\theta^2 \sqrt{1-\theta^2}} \operatorname{arc} \cosh \frac{-(1+\theta\eta)}{\theta+\eta} - \frac{\sqrt{1-\eta^2}}{\theta(\theta+\eta)} \right] \quad (72)$$

and for $0 < \eta < 1$

$$\frac{\partial w_{z=0}}{\partial \theta} = \frac{\beta C_1}{2\pi V_0} \left[\frac{1}{\theta^2} \operatorname{arc} \cosh \frac{1}{\eta} - \frac{1}{\theta^2 \sqrt{1-\theta^2}} \operatorname{arc} \cosh \frac{1+\theta\eta}{\theta+\eta} - \frac{\sqrt{1-\eta^2}}{\theta(\theta+\eta)} \right] \quad (73)$$

It can be verified easily that

$$\frac{\partial}{\partial \eta} \operatorname{arc} \cosh \frac{-1}{\eta} = \frac{-1}{\eta \sqrt{1-\eta^2}} \text{ for } \eta < 0$$

and

$$\frac{\partial}{\partial \eta} \operatorname{arc} \cosh \frac{1}{\eta} = \frac{-1}{\eta \sqrt{1-\eta^2}} \text{ for } \eta > 0$$

while

$$\frac{\partial}{\partial \eta} \operatorname{arc} \cosh \frac{1+\theta\eta}{\theta+\eta} = -\frac{1}{\theta+\eta} \sqrt{\frac{1-\theta^2}{1-\eta^2}} \text{ for } -\theta < \eta < 1$$

and

$$\frac{\partial}{\partial \eta} \operatorname{arc} \cosh \frac{-(1+\theta\eta)}{\theta+\eta} = -\frac{1}{\theta+\eta} \sqrt{\frac{1-\theta^2}{1-\eta^2}} \quad \text{for } -1 < \eta < -\theta$$

Combining these results, the desired expressions result:

For $\eta < 0$

$$\begin{aligned} \frac{\partial w_{z=0}}{\partial \theta} &= \frac{\beta C_1}{2\pi V_0} \left\{ \frac{-\sqrt{1-\eta^2}}{\theta(\theta+\eta)} + \int_{-1}^{\eta} \left[\frac{-1}{\theta^2 \eta_1 \sqrt{1-\eta_1^2}} + \frac{1}{\theta^2 (\theta+\eta_1) \sqrt{1-\eta_1^2}} \right] d\eta_1 \right\} \\ &= \frac{-\beta C_1}{2\pi V_0} \left[\frac{\sqrt{1-\eta^2}}{\theta(\theta+\eta)} + \int_{-1}^{\eta} \frac{d\eta_1}{\theta \eta_1 (\theta+\eta_1) \sqrt{1-\eta_1^2}} \right] \end{aligned} \quad (74)$$

and for $\eta > 0$

$$\frac{\partial w_{z=0}}{\partial \theta} = \frac{-\beta C_1}{2\pi V_0} \left[\frac{\sqrt{1-\eta^2}}{\theta(\theta+\eta)} + \int_1^{\eta} \frac{d\eta_1}{\theta \eta_1 (\theta+\eta_1) \sqrt{1-\eta_1^2}} \right] \quad (75)$$

Summing the elements over the type 1 triangular wing, induced vertical velocity is

$$w(\eta)_{z=0} = \frac{-\beta}{2\pi V_0} \int_0^{\theta_0} C(\theta) \left[\frac{\sqrt{1-\eta^2}}{\theta(\theta+\eta)} + \int_{-1}^{\eta} \frac{d\eta_1}{\theta \eta_1 (\theta+\eta_1) \sqrt{1-\eta_1^2}} \right] d\theta \quad (76)$$

where $C(\theta) = \phi_u - \phi_l$ for the element at $\theta = \arctan \frac{\theta}{\beta}$. If $w_{z=0}$ is constant, then

$$\frac{\partial w_{z=0}}{\partial \eta} = 0$$

and from this criterion the function $C(\theta)$ will be determined.

Thus

$$0 = \frac{-\beta}{2\pi V_0} \left[\frac{\partial}{\partial \eta} \int_0^{\theta_0} \frac{\sqrt{1-\eta^2} C(\theta)}{\theta(\theta+\eta)} d\theta + \frac{\partial}{\partial \eta} \int_0^{\theta_0} C(\theta) d\theta \int_{-1}^{\eta} \frac{d\eta_1}{\theta \eta_1 (\theta + \eta_1) \sqrt{1-\eta_1^2}} \right]$$

and, after introduction of the function $G_2(\eta, \theta)$ where

$$G_2(\eta, \theta) = \int_{-1}^{\eta} \frac{d\eta_1}{\theta \eta_1 (\theta + \eta_1) \sqrt{1-\eta_1^2}}$$

the integral equation may be written in the form

$$0 = \frac{\partial}{\partial \eta} \sqrt{1-\eta^2} \int_0^{\theta_0} \frac{C(\theta) d\theta}{\theta(\theta+\eta)} + \lim_{\epsilon \rightarrow 0} \left[\int_0^{-\eta-\epsilon} C(\theta) \frac{\partial G_2}{\partial \eta} d\theta + \int_{-\eta+\epsilon}^{\theta_0} C(\theta) \frac{\partial G_2}{\partial \eta} d\theta \right] \\ + \lim_{\epsilon \rightarrow 0} \left[-C(-\eta-\epsilon) G_2(\eta, -\eta-\epsilon) + C(-\eta+\epsilon) G_2(\eta, -\eta+\epsilon) \right] \quad (77)$$

The function $G_2(\eta, \theta)$ is given by the first two terms in the right-hand members of equations (71) and (72) and from consideration of these terms it follows that, provided $C(\theta)$ is a continuous function, equation (77) becomes

$$0 = \frac{\partial}{\partial \eta} \sqrt{1-\eta^2} \int_0^{\theta_0} \frac{C(\theta) d\theta}{\theta(\theta+\eta)} + \frac{1}{\eta \sqrt{1-\eta^2}} \int_0^{\theta_0} \frac{C(\theta) d\theta}{\theta(\theta+\eta)} \quad (78)$$

Comparison of equations (52) and (78), together with equation (53), indicates that the function $C(\theta)$ is of the form

$$C(\theta) = C_1 \sqrt{\frac{\theta}{\theta_0 - \theta}} \quad (79)$$

where C_1 is a constant. Substituting from equation (79) into equation (76).

$$\frac{2\pi V_{Oz=0}}{\beta C_1} = - \int_0^{\theta_0} \frac{\sqrt{1-\eta^2} d\theta}{(\theta+\eta) \sqrt{\theta(\theta_0-\theta)}} - \int_0^{\theta_0} \frac{d\theta}{\sqrt{\theta(\theta_0-\theta)}} \int_{-1}^{\eta} \frac{d\eta_1}{\eta_1 \sqrt{1-\eta_1^2} (\theta+\eta_1)} \quad (80)$$

The region of integration in the η_1, θ plane for the double integral of equation (80) is shown in figure 14 for the case in which $-\theta_0 < \eta < 0$. A singularity in the integrand of the double integral exists along the line $\theta + \eta_1 = 0$. Reversing the order of integration in the double integral, equation (80) may be rewritten as

$$\begin{aligned} \frac{2\pi V_{Oz=0}}{\beta C_1} = & - \sqrt{1-\eta^2} \int_0^{\theta_0} \frac{d\theta}{(\theta+\eta) \sqrt{\theta(\theta_0-\theta)}} \\ & - \int_{-1}^{-\theta_0} \frac{d\eta_1}{\eta_1 \sqrt{1-\eta_1^2}} \int_0^{\theta_0} \frac{d\theta}{(\theta+\eta_1) \sqrt{\theta(\theta_0-\theta)}} - \int_{-\theta_0}^{\eta} \frac{d\eta_1}{\eta_1 \sqrt{1-\eta_1^2}} \int_0^{\theta_0} \frac{d\theta}{(\theta+\eta_1) \sqrt{\theta(\theta_0-\theta)}} \end{aligned} \quad (81)$$

The single integral in equation (81) has a singularity at $\theta = -\eta$ since $-\theta_0 < \eta < 0$ and η therefore lies inside the region of integration. A corresponding singularity occurs in the second of the double integrals at $\theta = -\eta_1$. Consider, therefore, the integral

$$\int_0^{\theta_0} \frac{d\theta}{(\theta+\eta) \sqrt{\theta(\theta_0-\theta)}} = \lim_{\epsilon \rightarrow 0} \left[\int_0^{-\eta-\epsilon} \frac{d\theta}{(\theta+\eta) \sqrt{\theta_0-\theta}} + \int_{-\eta+\epsilon}^{\theta_0} \frac{d\theta}{(\theta+\eta) \sqrt{\theta(\theta_0-\theta)}} \right] \quad (82)$$

The indefinite integral is

$$\frac{1}{\sqrt{-\eta\theta_0-\eta^2}} \ln \frac{-\eta\theta_0+\theta\theta_0+2\eta\theta - 2\sqrt{(-\eta\theta_0-\eta^2)\theta(\theta_0-\theta)}}{\theta+\eta}$$

so that the definite integral is

$$\lim_{\epsilon \rightarrow 0} \frac{1}{\sqrt{-\eta\theta_0 - \eta^2}} \left\{ \ln \frac{\theta_0}{-\theta_0} + \ln \frac{[-2\eta\theta_0 - \epsilon\theta_0 - 2\eta^2 - 2\epsilon\eta - 2\sqrt{(-\eta\theta_0 - \eta^2)(-\eta - \epsilon)(\theta_0 + \eta + \epsilon)}]}{[-2\eta\theta_0 + \epsilon\theta_0 - 2\eta^2 + 2\epsilon\eta - 2\sqrt{(-\eta\theta_0 - \eta^2)(-\eta + \epsilon)(\theta_0 + \eta - \epsilon)}]} \right\} \frac{\epsilon}{(-\epsilon)}$$

The value of this expression is 0 and equation (81) can therefore be written as

$$\frac{2\pi V_0 w_{z=0}}{\beta C_1} = - \int_{-1}^{-\theta_0} \frac{d\eta_1}{\eta_1 \sqrt{1-\eta_1^2}} \int_0^{\theta_0} \frac{d\theta}{(\theta + \eta_1) \sqrt{\theta(\theta_0 - \theta)}} \quad (83)$$

Since, in this region of integration, $-1 < \eta < -\theta_0$ it follows that

$$\begin{aligned} \int_0^{\theta_0} \frac{d\theta}{(\theta + \eta_1) \sqrt{\theta(\theta_0 - \theta)}} &= \frac{1}{\sqrt{\eta_1\theta_0 + \eta_1^2}} \arctan \frac{-\eta_1\theta_0 + \theta\theta_0 + 2\eta_1\theta}{2\sqrt{(\eta_1^2 + \eta_1\theta_0)\theta(\theta_0 - \theta)}} \Big|_0^{\theta_0} \\ &= \frac{-\pi}{\sqrt{\eta(\theta_0 + \eta)}} \end{aligned}$$

and

$$w_{z=0} = \frac{\beta C_1}{2V_0} \int_{-1}^{-\theta_0} \frac{d\eta_1}{\eta_1 \sqrt{1-\eta_1^2} \sqrt{\eta_1(\eta_1 + \theta_0)}} \quad (84)$$

The integral of equation (84) can be transformed by means of purely algebraic substitutions into a form which integrates immediately into complete elliptic integrals of the first and second kind. However, in the consideration of the type 3 plan form it will be necessary to resort to other methods of transformation, so that a more uniform approach, employing Jacobian elliptic functions, will be used throughout. Reference 12 contains a complete discussion of these functions.

The quartic under the radical in equation (84) can be reduced to an expression of the type appearing in elliptic integrals of canonical form. This is accomplished by successive application of the transformations

$$\eta_1 = -\frac{k+lS}{1+S} \quad \text{and} \quad S = \frac{k}{t}$$

where k and l are chosen so as to destroy the odd powers of the variable. By means of these transformations, induced velocity becomes

$$w_{z=0} = \frac{\beta C_1}{2V_0} \frac{(1-k^2)k^2}{\sqrt{k(\theta_0-k)(1-k^2)}} \int_1^{\frac{1}{k}} \frac{[t^2 + (k - \frac{l}{k})t - l] dt}{(1-k^2t^2) \sqrt{(1-t^2)(k^2t^2-1)}} \quad (85)$$

where

$$k = \frac{1 - \sqrt{1 - \theta_0^2}}{\theta_0}$$

The integration of equation (85) will be performed after first considering two parts such that $w_{z=0} = w_1 + w_2$ where

$$w_1 = \frac{\beta C_1}{2V_0} \frac{(1-k^2)k^2}{\sqrt{k(\theta_0-k)(1-k^2)}} \int_1^{\frac{1}{k}} \frac{(k - \frac{l}{k})t dt}{(1-k^2t^2) \sqrt{(1-t^2)(k^2t^2-1)}}$$

and

$$w_2 = -\frac{\beta C_1}{2V_0} \frac{(1-k^2)k^2}{\sqrt{k(\theta_0-k)(1-k^2)}} \int_1^{\frac{1}{k}} \frac{(1-t^2) dt}{(1-k^2t^2) \sqrt{(1-t^2)(k^2t^2-1)}}$$

This separation is prompted by the fact that the integral for w_1 is expressible in terms of elementary functions after the simple transformation $t^2=z$. The results of such an integration lead to a value which is zero at the lower limit and infinite at the upper limit.

However, an inspection of the original integral in equation (84) shows that $w_{z=0}$ is finite so the infinity obtained for w_1 must be canceled by a corresponding infinity of equal magnitude in w_2 . The actual proof of this statement necessitates, of course, treating the combined expressions as an indeterminate form where the upper limits of the integrals for w_1 and w_2 are replaced by $\frac{1}{k} + \epsilon$ and the limit is taken as ϵ approaches zero.

Introduce now in the integration of w_2 Jacobian elliptic functions and set

$$t = \operatorname{sn}(u, k) = \operatorname{sn} u$$

so that

$$dt = \operatorname{cn} u \operatorname{dn} u \, du$$

The expression for w_2 becomes

$$w_2 = \frac{\beta C_1}{2V_0} \frac{(1-k^2)k^2 i}{\sqrt{k(\theta_0 - k)(1-k^2)}} \int_K^{K+iK'} \frac{\operatorname{cn}^2 u}{\operatorname{dn}^2 u} \, du$$

where K and K' are the complete elliptic integrals of the first kind with respective moduli k and $k' = \sqrt{1-k^2}$. Integrating and combining with w_1 , it follows that

$$w_{z=0} = w_1 + \frac{\beta C_1}{2V_0} \frac{(1-k^2)k^2 i}{\sqrt{k(\theta_0 - k)(1-k^2)}} \left[\frac{1}{k^2} \left(u + k^2 \frac{\operatorname{sn} u \operatorname{cn} u}{\operatorname{dn} u} - E(u) \right) \right]_K^{K+iK'}$$

where $E(u)$ is the incomplete elliptic integral of the second kind. After substitution of the limits,

$$w_{z=0} = - \frac{\beta C_1}{2V_0} \frac{(1-k^2)E'}{\sqrt{k(\theta_0 - k)(1-k^2)}} \quad (86)$$

where E' is the complete elliptic integral of the second kind with modulus $k' = \sqrt{1-k^2}$. Equation (86) can be further simplified by writing k in terms of θ_0 so that

$$w_{z=0} = -\frac{\beta C_1}{2V_0} \sqrt{\frac{2[1+\sqrt{1-\theta_0^2}]}{\theta_0^2}} E' \quad (87)$$

where the modulus of E' is $\sqrt{1-k^2}$ and $k = \frac{1-\sqrt{1-\theta_0^2}}{\theta_0}$

For the loading in question

$$p_l - p_u = \rho_0 (\phi_u - \phi_l) = \rho_0 C_1 \sqrt{\frac{\theta}{\theta_0 - \theta}}$$

and

$$\frac{p_l - p_u}{\frac{1}{2}\rho_0 V_0^2} = \frac{\Delta p}{q} = \frac{2C_1}{V_0^2} \sqrt{\frac{\theta}{\theta_0 - \theta}}$$

By means of equation (87) the constant C_1 may be eliminated and, since $\alpha = -w_{z=0}/V_0$,

$$\frac{\Delta p}{q} = \frac{2\alpha}{\beta E'} \sqrt{\frac{2\theta(1-\sqrt{1-\theta_0^2})}{\theta_0 - \theta}} \quad (88)$$

Figure 15 shows the variation of $\frac{\beta}{\alpha} \frac{\Delta p}{q}$ with $\beta \tan \delta$ for values of $\beta \tan \delta_0$ equal to 0.3, 0.6, and 0.9.

From equations (88) and (59) the lift coefficient of a right triangle wing can be determined. It follows that

$$C_L = \frac{\pi\alpha}{\beta E'} \sqrt{2(1-\sqrt{1-\theta_0^2})} \quad (89)$$

Since the aspect ratio A is given by the equation $A = 2\theta_0/\beta$, equation (89) can be used to find the lift curve slope as a function of A . In figure 16 a plot of $\beta \frac{dC_L}{d\alpha}$ as a function of βA is given.

Triangular plan form, type 2.— Figure 12(b) shows the symmetrical type of triangular plan form considered in this section. The semi-vertex angle is δ_0 and θ_0 is defined by the relation

$$\theta_0 = \beta \tan \delta_0$$

The loading element used in the previous section can be used again and equations (74) and (75) are applicable directly. Because of the symmetry of the figure, it is necessary merely to insure the constancy of $w_{z=0}$ over the left half of the wing in order that the entire wing be specified a flat plate.

When the lifting elements are on the left portion of the figure, the induced vertical velocity contributed by them will be given by equation (76) for, aside from the change in the loading function $C(\theta)$, this situation corresponds exactly to the condition considered in the type 1 plan form. To equation (76) must also be added the expression for the induced vertical velocity on the left side of the wing produced by the lifting elements on the right. In the development of equations (74) and (75) the element was situated in the second quadrant of the axial system; that is, y was negative and x was positive. For an element on the right side of the wing, equations (74) and (75) are again applicable if the axial system is changed so that the Y -axis reverses its direction while the X -axis remains fixed. Since, by definition,

$$\eta = \beta \frac{y}{x}$$

it follows that the change of sign for y results in a change of sign for η . When the lifting element is on the right-hand side of the wing the expressions for gradient of induced vertical velocity are, thus,

for $\eta < 0$

$$\frac{\partial w_{z=0}}{\partial \theta} = - \frac{\beta C_1}{2\pi V_0} \left[\frac{\sqrt{1-\eta^2}}{\theta(\theta-\eta)} + \int_{-1}^{\eta} \frac{d\eta_1}{\theta\eta_1(\theta-\eta_1)\sqrt{1-\eta_1^2}} \right] \quad (90)$$

and for $\eta > 0$

$$\frac{\partial w_{z=0}}{\partial \theta} = -\frac{\beta C_1}{2\pi V_0} \left[\frac{\sqrt{1-\eta^2}}{\theta(\theta-\eta)} + \int_1^\eta \frac{d\eta_1}{\theta\eta_1(\theta-\eta_1)\sqrt{1-\eta_1^2}} \right] \quad (91)$$

Summing the elements over the type 2 triangular wing, induced vertical velocity over the portion of the wing for which $-1 < \eta < 0$ is

$$\begin{aligned} w(\eta)_{z=0} &= -\frac{\beta}{2\pi V_0} \int_0^{\theta_0} \left[\frac{\sqrt{1-\eta^2}}{\theta(\theta+\eta)} + \int_{-1}^\eta \frac{d\eta_1}{\theta\eta_1(\theta+\eta_1)\sqrt{1-\eta_1^2}} \right] C(\theta) d\theta \\ &\quad - \frac{\beta}{2\pi V_0} \int_0^{\theta_0} \left[\frac{\sqrt{1-\eta^2}}{\theta(\theta-\eta)} + \int_{-1}^\eta \frac{d\eta_1}{\theta\eta_1(\theta-\eta_1)\sqrt{1-\eta_1^2}} \right] C(\theta) d\theta \\ &= -\frac{2\beta}{2\pi V_0} \int_0^{\theta_0} \left[\frac{\sqrt{1-\eta^2}}{\theta^2-\eta^2} + \int_{-1}^\eta \frac{d\eta_1}{\eta_1(\theta^2-\eta_1^2)\sqrt{1-\eta_1^2}} \right] C(\theta) d\theta \quad (92) \end{aligned}$$

The function $C(\theta)$ in equation (92) must give a constant value for $w(\eta)_{z=0}$ so that $\partial w_{z=0}/\partial \eta$ will vanish and also be such a function that the loading will be symmetrical about the line $\theta=0$. Imposing these conditions it can be shown that a solution is given by the relation

$$C(\theta) = \frac{C_1}{\sqrt{\theta_0^2 - \theta^2}} \quad (93)$$

and, after substitution in equation (92),

$$\frac{\pi V_0 w_{z=0}}{\beta C_1} = -\int_0^{\theta_0} \frac{\sqrt{1-\eta^2}}{(\theta^2-\eta^2)\sqrt{\theta_0^2-\theta^2}} d\theta - \int_0^{\theta_0} \frac{d\theta}{\sqrt{\theta_0^2-\theta^2}} \int_{-1}^\eta \frac{d\eta_1}{\eta_1\sqrt{1-\eta_1^2}(\theta^2-\eta_1^2)} \quad (94)$$

The integration of equation (94) is to be performed under the assumption that $-\theta_0 < \eta < 0$ so that the region of integration in the η_1, θ plane for the double integral is as shown in figure 14. A singularity of the integrand exists along the line $\theta + \eta_1 = 0$. Reversing the order of integration in the double integral, equation (94) may be written in the form

$$\frac{\pi V_{\infty} w_{z=0}}{\beta C_1} = -\sqrt{1-\eta^2} \int_0^{\theta_0} \frac{d\theta}{(\theta^2 - \eta^2) \sqrt{\theta_0^2 - \theta^2}} \quad (95)$$

$$- \int_{-1}^{-\theta_0} \frac{d\eta_1}{\eta_1 \sqrt{1-\eta_1^2}} \int_0^{\theta_0} \frac{d\theta}{(\theta^2 - \eta_1^2) \sqrt{\theta_0^2 - \theta^2}} - \int_{-\theta_0}^{\eta} \frac{d\eta_1}{\eta_1 \sqrt{1-\eta_1^2}} \int_0^{\theta_0} \frac{d\theta}{(\theta^2 - \eta_1^2) \sqrt{\theta_0^2 - \theta^2}}$$

Evaluation of integrals of the form

$$I = - \int_0^{\theta_0} \frac{d\theta}{(\theta^2 - \eta^2) \sqrt{\theta_0^2 - \theta^2}}$$

can be accomplished by means of the substitution

$$x = \frac{\theta}{\theta_0}$$

After substitution, the integral becomes

$$I = \frac{1}{\theta_0^2} \int_0^1 \frac{dx}{\left(\frac{\eta^2}{\theta_0^2} - x^2\right) \sqrt{1-x^2}}$$

and, by straightforward integration,

$$I = \begin{cases} 0 & \text{for } \frac{\eta^2}{\theta_0^2} < 1 \\ \frac{\pi}{2\eta \sqrt{\eta^2 - \theta_0^2}} & \text{for } \frac{\eta^2}{\theta_0^2} > 1 \end{cases}$$

This result shows that the second double integral of equation (95) vanishes (since $\theta_o^2 > \eta_1^2$) as does also the single integral in the equation. For the remaining double integral, however, $\theta_o^2 < \eta_1^2$ and

$$w_{z=0} = \frac{\beta C_1}{2V_o} \int_{-1}^{-\theta_o} \frac{d\eta_1}{\eta_1^2 \sqrt{1-\eta_1^2} \sqrt{\eta_1^2 - \theta_o^2}} \quad (96)$$

Setting

$$z = -\frac{1}{\eta_1}$$

equation (96) transforms to

$$w_{z=0} = -\frac{\beta C_1}{2V_o} \int_1^{\frac{1}{\theta_o}} \frac{z^2 dz}{\sqrt{(z^2-1)(1-\theta_o^2 z^2)}}$$

Introducing the modulus $k = \theta_o$ and making the substitution

$$z = \operatorname{sn}(u, k) = \operatorname{sn} u$$

the expression for $w_{z=0}$ becomes

$$\begin{aligned} w_{z=0} &= \frac{i\beta C_1}{2V_o} \int_K^{K+iK'} \operatorname{sn}^2 u \, du \\ &= \frac{i\beta C_1}{2V_o k^2} [u - E(u)]_K^{K+iK'} \\ &= -\frac{\beta C_1}{2V_o k^2} E' \end{aligned}$$

where the prime again refers to the complementary modulus $k' = \sqrt{1-k^2}$ of the complete elliptic integral. Since $k = \theta_o$

$$w_{z=0} = -\frac{\beta C_1}{2V_o \theta_o^2} E' \quad (97)$$

where $k' = \sqrt{1-\theta_0^2}$

For the loading in question

$$\frac{p_l - p_u}{\frac{1}{2}\rho_0 V_0^2} = \frac{\Delta p}{q} = \frac{2C_1}{V_0^2 \sqrt{\theta_0^2 - \theta^2}}$$

so that, eliminating C_1 between this equation and equation (97),

$$\frac{\Delta p}{q} = \frac{4\alpha\theta_0^2}{\beta \sqrt{\theta_0^2 - \theta^2} E'} \quad (98)$$

Figure 17 shows the variation of $\frac{\beta}{\alpha} \frac{\Delta p}{q}$ with $\beta \tan \delta$ for values of $\beta \tan \delta_0$ equal to 0.3, 0.6, and 0.9.

Using equations (98) and (59) it is possible to find the expression for lift coefficient of a triangular or delta wing. Thus:

$$C_L = \frac{2\pi\theta_0\alpha}{\beta E'} \quad (99)$$

Since aspect ratio of the wing is

$$A = \frac{4\theta_0}{\beta}$$

lift coefficient becomes

$$C_L = \frac{\pi\alpha A}{2E'} \quad (100)$$

where the modulus of E' is $k' = \sqrt{1 - \frac{A^2\beta^2}{16}}$. This result agrees with that obtained in another manner by Stewart (reference 8). In figure 18 a plot is given of $\beta \frac{dC_L}{d\alpha}$ as a function of βA .

Triangular plan form, type 3. Figure 12(c) shows the plan form now to be considered. Relative to the X-axis or free-stream direction the sides of the triangle form the angles δ_0 and δ_1 so that the total vertex angle is $\delta_0 + \delta_1 = 2\Delta$. The variables θ_0 and θ_1 are also introduced satisfying the relations $\theta_0 = \beta \tan \delta_0$, $\theta_1 = \beta \tan \delta_1$. The same loading element which was used for type 1 and type 2 triangles may be used and equations (74) and (75) apply. It will then be necessary to determine the distribution of load so

that the induced vertical velocity over the plan form is a constant. Since this induced velocity must be the same on both sides of the $\delta=0$ axis, two equations result:

for $-1 < \eta < 0$

$$w_{z=0} = -\frac{\beta}{2\pi V_0} \int_{-\theta_1}^{\theta_0} \left[\frac{\sqrt{1-\eta^2}}{\theta(\theta+\eta)} + \int_{-1}^{\eta} \frac{d\eta_1}{\theta\eta_1(\theta+\eta_1)\sqrt{1-\eta_1^2}} \right] C(\theta) d\theta \quad (101)$$

for $0 < \eta < 1$

$$w_{z=0} = -\frac{\beta}{2\pi V_0} \int_{-\theta_1}^{\theta_0} \left[\frac{\sqrt{1-\eta^2}}{\theta(\theta+\eta)} + \int_1^{\eta} \frac{d\eta_1}{\theta\eta_1(\theta+\eta_1)\sqrt{1-\eta_1^2}} \right] C(\theta) d\theta \quad (102)$$

From the solutions to the problems of type 1 and type 2 it is possible to construct a solution of the more general problem by expressing the loading function in the form

$$C(\theta) = \frac{A\theta + B}{\sqrt{(\theta_1+\theta)(\theta_0-\theta)}} \quad (103)$$

where A and B are constants which can be determined in terms of $w_{z=0}$ from equations (101) and (102). Substituting from equation (103) into equation (101) and (102), the expressions for induced velocity become

$$\begin{aligned} w_{z=0} &= \frac{\beta}{2V_0} [AH_1(\theta_0, \theta_1) + BH_2(\theta_0, \theta_1)] \\ w_{z=0} &= \frac{\beta}{2V_0} [-AH_1(\theta_1, \theta_0) + BH_2(\theta_1, \theta_0)] \end{aligned} \quad (104)$$

where

$$H_1(\theta_1, \theta_0) = \int_1^{\theta_1} \frac{d\eta_1}{\eta_1 \sqrt{(1-\eta_1^2)(\eta_1-\theta_1)(\eta_1+\theta_0)}} \quad (105)$$

$$H_2(\theta_1, \theta_0) = \int_1^{\theta_1} \frac{d\eta_1}{\eta_1^2 \sqrt{(1-\eta_1^2)(\eta_1-\theta_1)(\eta_1+\theta_0)}} \quad (106)$$

The evaluation of $H_1(\theta_1, \theta_0)$ and $H_2(\theta_1, \theta_0)$ is accomplished in the same manner as has been used previously: first, a reduction of the quartic under the radical to canonical form and second, transformation by means of Jacobian elliptic functions followed by direct integration. Since the calculations for both equations are quite similar, only in the case of $H_1(\theta_1, \theta_0)$ will the details be mentioned. All relevant information can be found in reference 12.

Applying the transformations $\eta_1 = \frac{a+bS}{1+S}$ and $S = \frac{e}{t}$, where

$$a = \frac{1-\theta_0\theta_1 - \sqrt{(1-\theta_0^2)(1-\theta_1^2)}}{\theta_0 + \theta_1} \quad (107)$$

$$b = \frac{1-\theta_0\theta_1 + \sqrt{(1-\theta_0^2)(1-\theta_1^2)}}{\theta_0 + \theta_1} \quad (108)$$

and introducing the symbols R and k defined as

$$R = \frac{(b-a)}{\sqrt{(1-a^2)(\theta_0-a)(\theta_1+a)}} \quad (109)$$

$$k = \frac{1-\theta_0 b}{\theta_0 - b} \quad (110)$$

equation (105) reduces to

$$H_1 = a^2 R k \int_1^{\frac{1}{k}} \frac{[t^2 + t(a-b) - 1] dt}{(1-a^2 t^2) \sqrt{(t^2-1)(1-k^2 t^2)}} \quad (111)$$

The integrand divides naturally into two parts, one containing even powers and one containing odd powers of t in the numerator. The latter part integrates into elementary functions after substituting $t = u^2$ and equation (111) thereby becomes

$$H_1 = a^2 Rk \left[-\frac{\pi}{2} \sqrt{\frac{b^2-1}{k^2-a^2}} + \int_1^{\frac{1}{k}} \frac{(t^2-1)dt}{(1-a^2t^2) \sqrt{(t^2-1)(1-k^2t^2)}} \right] \quad (112)$$

Setting

$$I_2 = \int_1^{\frac{1}{k}} \frac{(t^2-1)dt}{(1-a^2t^2) \sqrt{(t^2-1)(1-k^2t^2)}}$$

and substituting

$$x = \operatorname{sn}(u, k) = \operatorname{sn} u$$

it follows that

$$I_2 = i \int_K^{K+iK'} \left[1 + (a^2-1) \frac{\operatorname{sn}^2 u}{1-a^2 \operatorname{sn}^2 u} \right] du$$

If $\operatorname{sn} \gamma = \frac{a}{k}$, I_2 now may be written as

$$I_2 = -K' - i \frac{\operatorname{dn} \gamma}{k^2 \operatorname{sn} \gamma \operatorname{cn} \gamma} \int_K^{K+iK'} \frac{k^2 \operatorname{sn} \gamma \operatorname{cn} \gamma \operatorname{dn} \gamma \operatorname{sn}^2 u}{1-k^2 \operatorname{sn}^2 \gamma \operatorname{sn}^2 u} du$$

or

$$I_2 = -K' - i \sqrt{\frac{b^2-1}{k^2-a^2}} \left[\Pi(u, \gamma) \right]_K^{K+iK'} \quad (113)$$

where $\Pi(u, \gamma)$ is the fundamental elliptic integral of the third kind.

The evaluation of $\Pi(u, \gamma)$ is best achieved by means of its expression in terms of theta functions and zeta functions. Thus

$$\Pi(u, \gamma) = \frac{1}{2} \log \frac{\Theta(u-\gamma)}{\Theta(u+\gamma)} + uZ(\gamma)$$

and, substituting the limits in equation (113),

$$\Pi(K+iK', \gamma) - \Pi(K, \gamma) = \frac{1}{2} \log \frac{\Theta(K+iK'-\gamma) \Theta(K+\gamma)}{\Theta(K-\gamma) \Theta(K+iK'+\gamma)} + iK'Z(\gamma)$$

The Theta functions are a class known as quasi-periodic, that is, they satisfy the relations

$$\Theta(u+2K) = \Theta(u)$$

$$\Theta(u+2iK') = -e^{\frac{\pi}{K}(K'-iu)} \Theta(u)$$

From this property, together with the fact that $\Theta(u)$ is an even function, it follows that

$$\frac{\Theta(K+iK'-\gamma) \Theta(K+\gamma)}{\Theta(K-\gamma) \Theta(K+iK'+\gamma)} = e^{\frac{i\pi\gamma}{k}}$$

Moreover, since

$$Z(\gamma) = E(\gamma) - \gamma \frac{E}{K}$$

there results

$$\Pi(K+iK', \gamma) - \Pi(K, \gamma) = iK'E(\gamma) - i\gamma \frac{K'E}{K} + i \frac{\pi\gamma}{2K}$$

and

$$I_2 = -K' + \sqrt{\frac{b^2-1}{k^2-a^2}} \left\{ \frac{\pi\gamma}{2K} + K' \left[E(\gamma) - \gamma \frac{E}{K} \right] \right\} \quad (114)$$

The expression for equation (112) can now be written

$$H_1 = a^2 Rk \sqrt{\frac{b^2-1}{k^2-a^2}} \left\{ \frac{\pi}{2} \left[\frac{\gamma}{K} - 1 \right] + K' \left[E(\gamma) - \gamma \frac{E}{K} \right] \right\} - a^2 Rk K' \quad (115)$$

where the moduli are k for the nonprimed functions and $k' = \sqrt{1-k^2}$ for the primed functions. By definition, $\gamma = \arcsin \frac{a}{K} = F\left(\frac{a}{K}, k\right)$

where F is the incomplete elliptic integral of the first kind with argument $\frac{a}{k}$ and modulus k .

Using the same notation, the equation for H_2 is as follows:

$$H_2 = \frac{kRb}{b^2k^2-1} \left\{ a^2(1-k^2)K' - (1-a^2)E' - a(1-k^2) \sqrt{\frac{b^2-1}{b^2k^2-1}} \left[K'E(\gamma) - \gamma \frac{EK'}{K} + \frac{\pi}{2} \left(\frac{\gamma}{K} - 1 \right) \right] \right\} \quad (116)$$

Formulas (115) and (116) can now be combined with equations (104) to give

$$A = - \frac{V_0 w_{z=0}}{\beta E'} (\theta_0 - \theta_1) \sqrt{\frac{2G}{\theta_0 + \theta_1}} \quad (117)$$

and

$$B = - \frac{V_0 w_{z=0}}{\beta E'} 2\theta_0\theta_1 \sqrt{\frac{2G}{\theta_0 + \theta_1}} \quad (118)$$

where

$$G = \frac{1 + \theta_0\theta_1 - \sqrt{(1-\theta_0^2)(1-\theta_1^2)}}{\theta_0 + \theta_1} \quad (119)$$

and E' is the complete elliptic integral of the second kind with modulus $\sqrt{1-G^2}$.

From equations (103), (117), and (118)

$$\frac{p_l - p_u}{\frac{1}{2}\rho_0 V_0^2} = \frac{\Delta p}{q} = \frac{2C(\theta)}{V_0^2} = \frac{2}{V_0^2} \frac{A\theta + B}{\sqrt{(\theta_1 + \theta)(\theta_0 - \theta)}} = \frac{2\alpha}{\beta E'} \sqrt{\frac{2G}{\theta_0 + \theta_1}} \left[\frac{(\theta_0 - \theta_1)\theta + 2\theta_0\theta_1}{\sqrt{(\theta_1 + \theta)(\theta_0 - \theta)}} \right] \quad (120)$$

It should be remarked that the slope of the loading curve is zero at $\theta = 0$. Figures 19(a), 19(b), 19(c) show the variation of $\frac{\beta}{\alpha} \frac{\Delta p}{q}$

with $\beta \tan \delta$ for values of $\beta \tan \delta_1$ equal to 0.3, 0.6, and 0.9, respectively and for $\beta \tan \delta_0$ equal to 0, 0.3, 0.6, and 0.9.

From equations (59) and (119) the lift coefficient for a type 3 plan form is obtainable. Two cases will be developed here: first, when the trailing edge of the wing is perpendicular to the stream direction; second, when the trailing edge of the wing is perpendicular to the line of symmetry. The first configuration may be referred to as a skewed wing while the second configuration may be referred to as a symmetrical delta wing at an angle of sideslip. There results for the skewed wing then

$$C_L = \frac{\pi \alpha}{E' \beta} \sqrt{(\theta_0 + \theta_1) 2G} \quad (121)$$

where G is given by equation (119) and E' has the modulus $\sqrt{1-G^2}$. This result agrees with that given by R. C. Roberts in an abstract in reference 13.

For the more practical case of the delta wing at an angle of sideslip, figure 12(c), the lift coefficient can be expressed as

$$C_L = \frac{2\alpha\pi}{E'} \sec \Lambda \sqrt{\frac{G \tan \Delta}{\beta}} \quad (122)$$

where Λ is the angle of sideslip and 2Δ the angle between the leading edges, E' still has the modulus $\sqrt{1-G^2}$, and G is expressed in terms of θ_0 and θ_1 which are, in turn expressed in terms of Λ and Δ by the following equations

$$\left. \begin{aligned} \theta_0 &= \beta \tan (\Delta + \Lambda) \\ \theta_1 &= \beta \tan (\Delta - \Lambda) \end{aligned} \right\} \quad (123)$$

It should be noted that since the pressure distribution has been computed only for wings with leading edges behind the Mach cone springing from the apex and with a trailing edge ahead of the Mach

cones springing from the wing tips, formula (122) is valid only for cases where

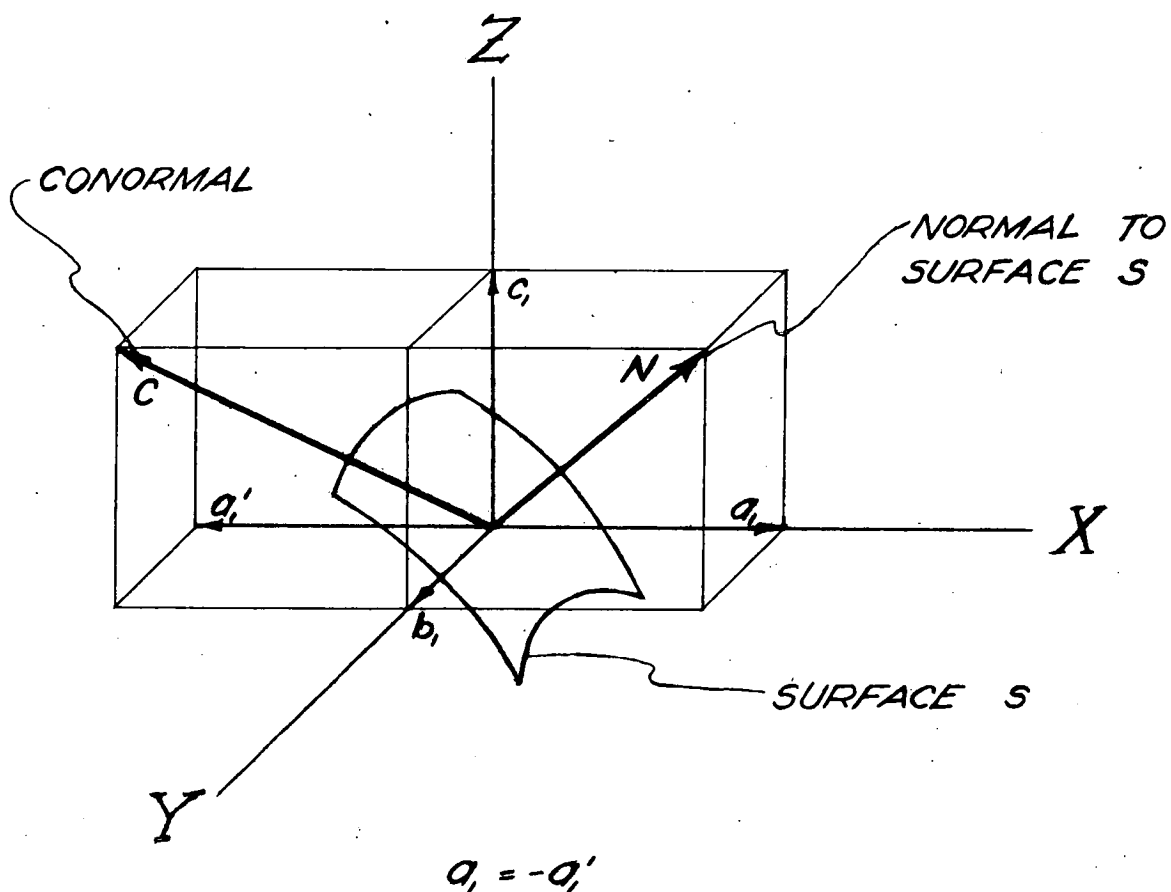
$$\left. \begin{array}{l} \mu + \Lambda < 90^\circ \\ \Delta + \Lambda < \mu \\ \Delta - \Lambda > 0 \end{array} \right\} \quad (124)$$

These restrictions are practically always met, however, for angles of sideslip which are likely to be encountered in flight.

Equation (122) is plotted in figure 20 where $\beta \frac{dC_L}{d\alpha}$ is shown as a function of sideslip and Δ . The figure shows that up to 15° of sideslip $\beta \frac{dC_L}{d\alpha}$ remains practically constant.

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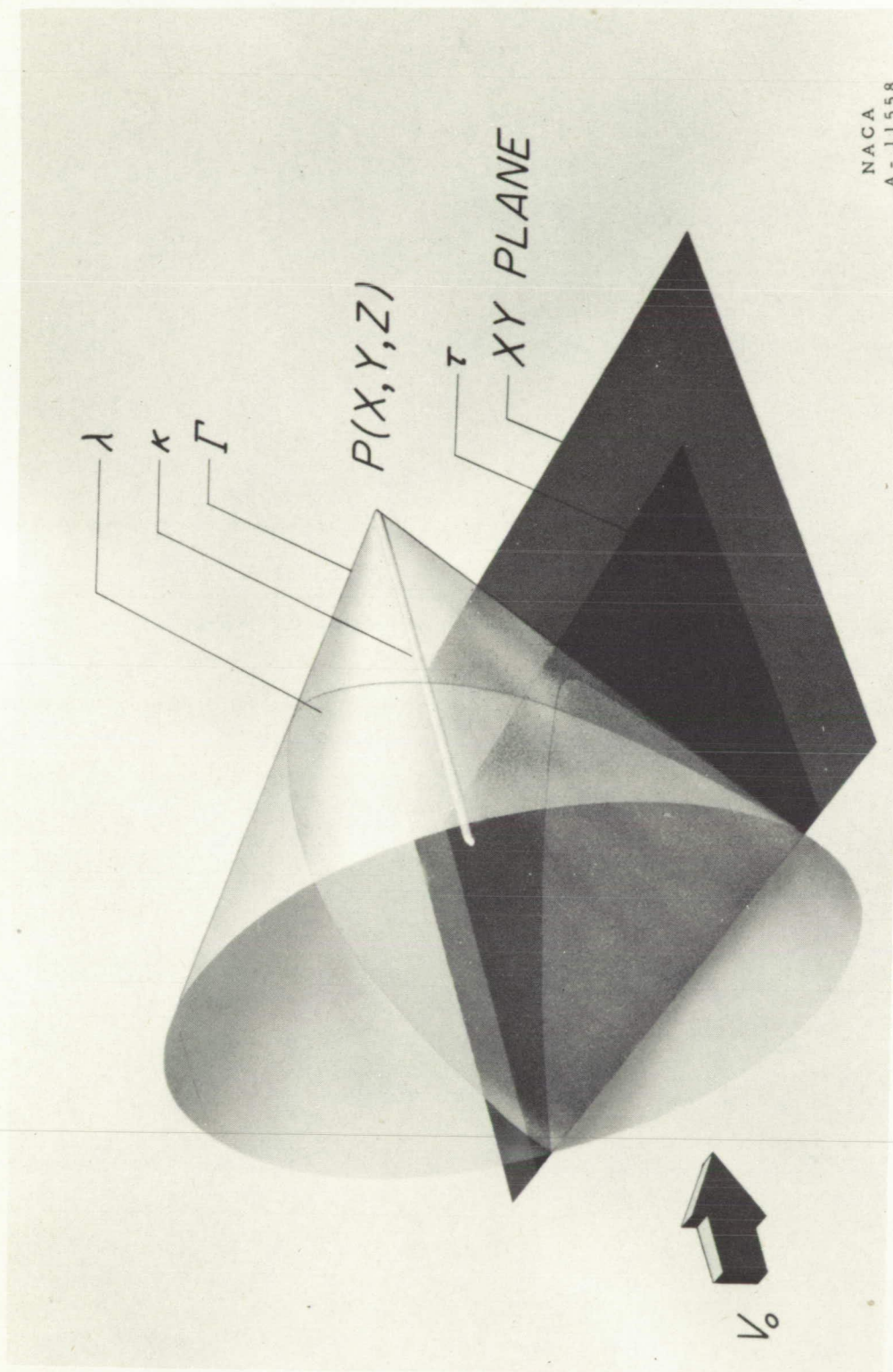
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$$\begin{aligned}(nX) &= \cos^{-1} a_i/N = \pi - \cos^{-1} q_i/C = \pi - (\nu X) \\(nY) &= \cos^{-1} b_i/N = \cos^{-1} b_i/C = (\nu Y) \\(nZ) &= \cos^{-1} c_i/N = \cos^{-1} c_i/C = (\nu Z)\end{aligned}$$

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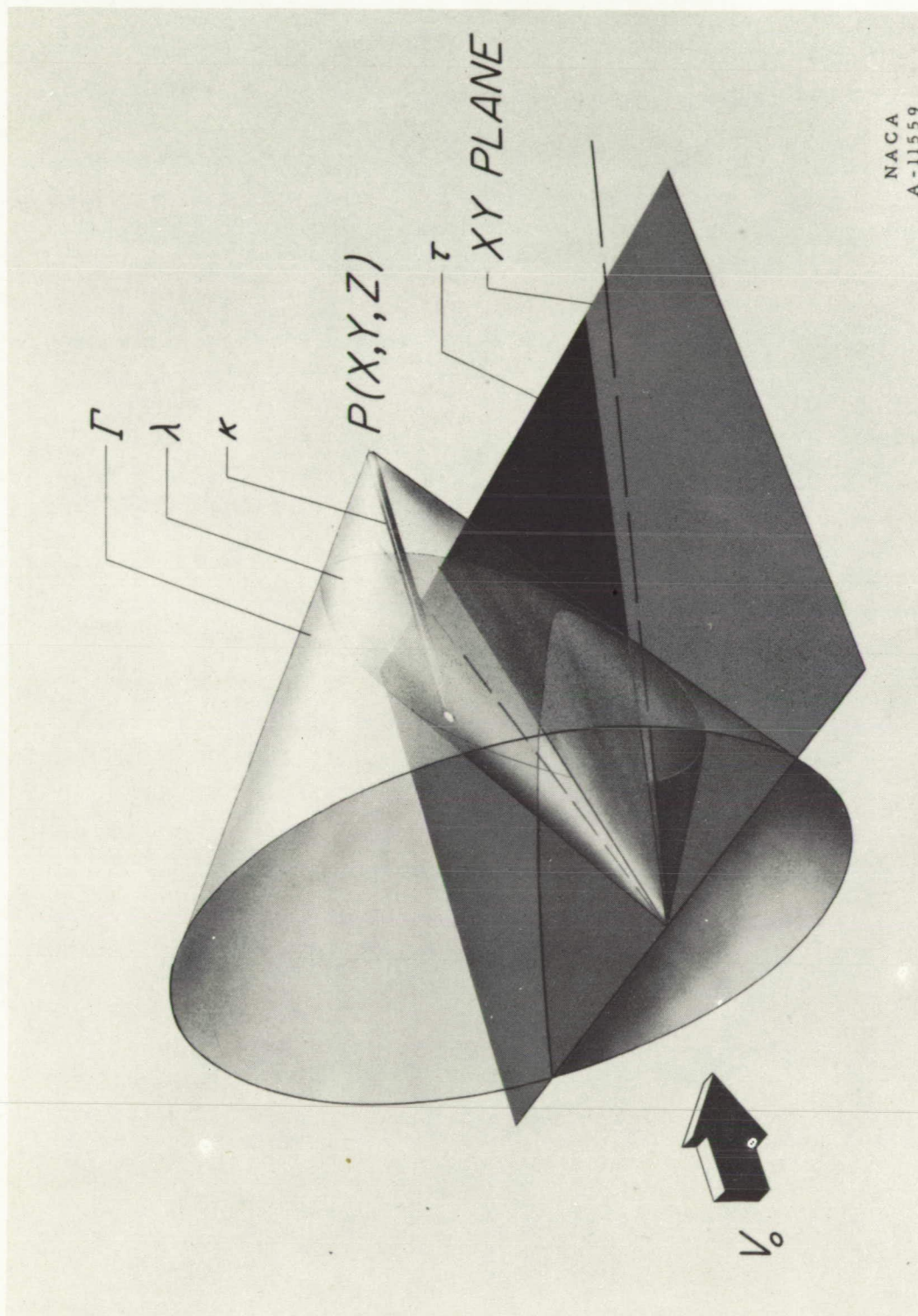
FIGURE 1.- THE GEOMETRIC RELATIONS BETWEEN NORMAL AND CONORMAL TO SURFACE S.



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(a) Rectangular plan form.

Figure 2.- Mach forecone from point $P(X, Y, Z)$ intersecting surface τ .



(b) Triangular plan form.

Figure 2.- Concluded.

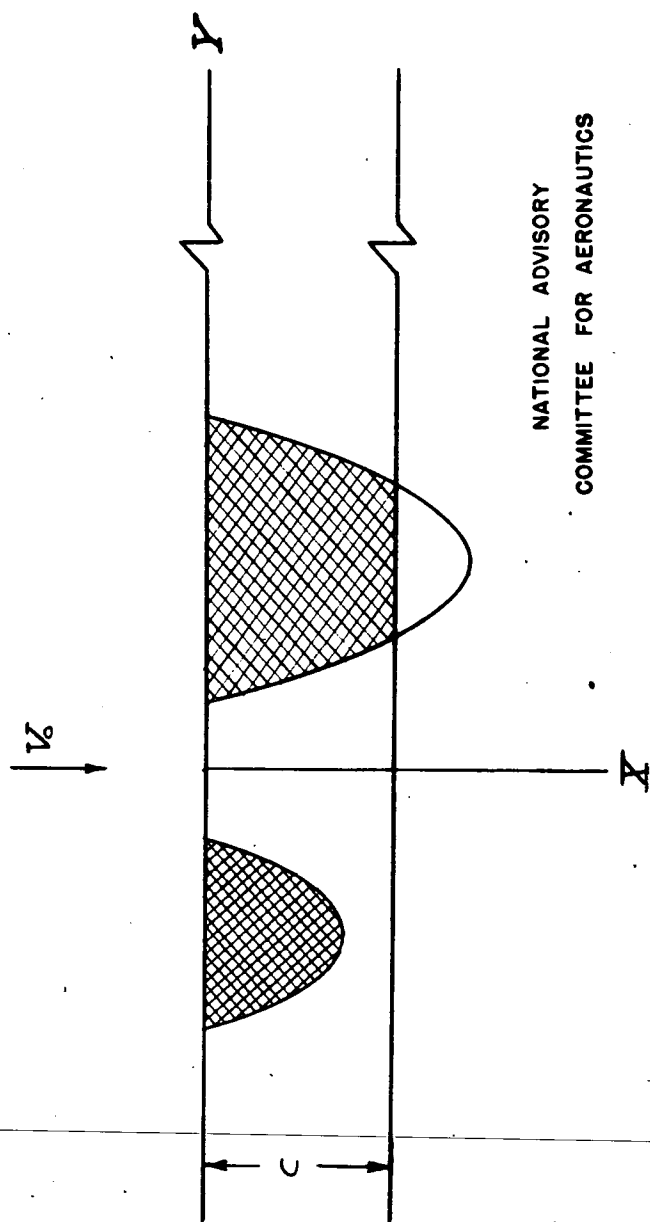


FIGURE 3.— REGIONS OF INTEGRATION FOR INFINITE SPAN UNSWEPT WING.

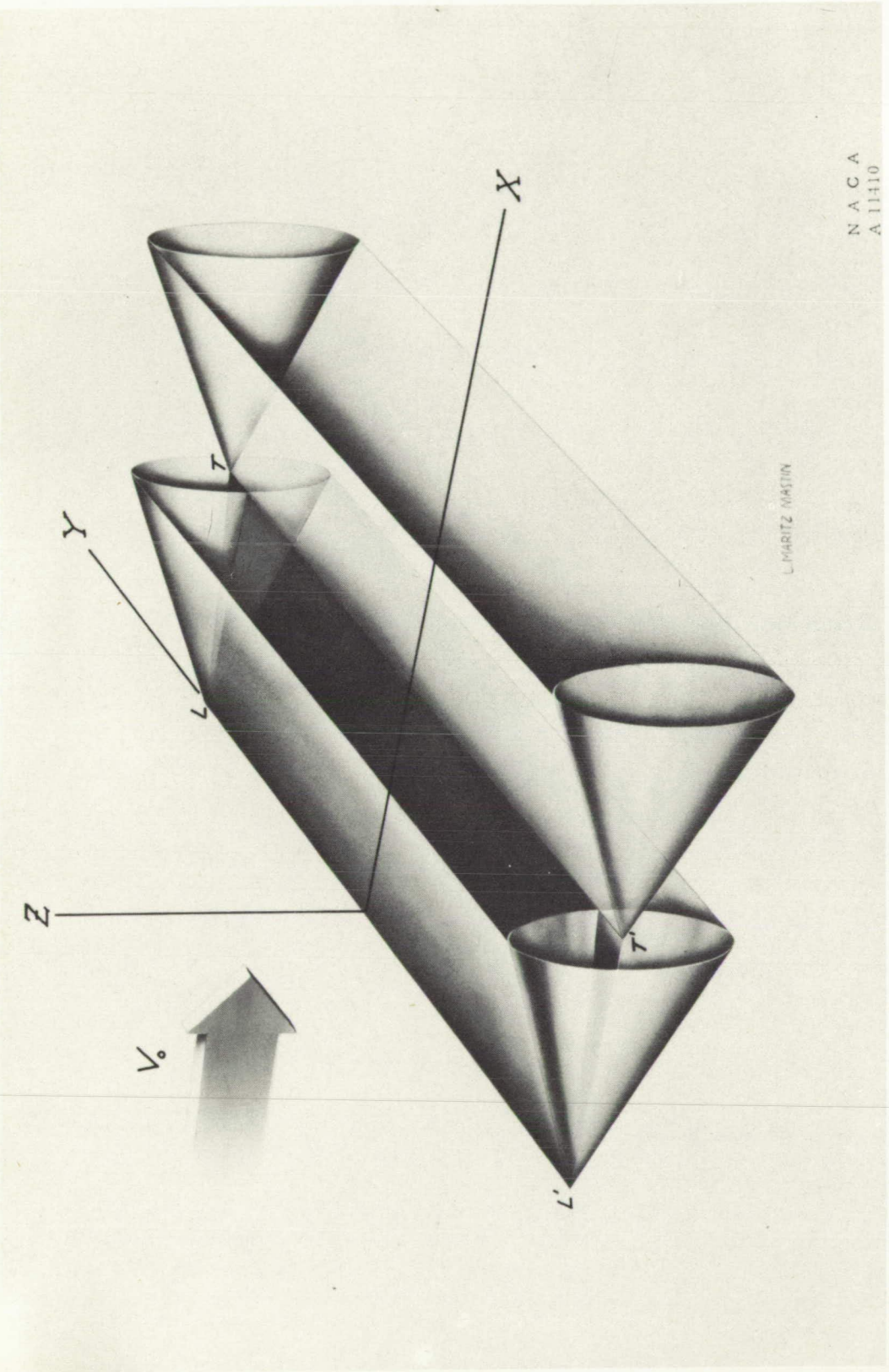


Figure 4.- Lifting surface with Mach cones and coordinate system for rectangular plan form.

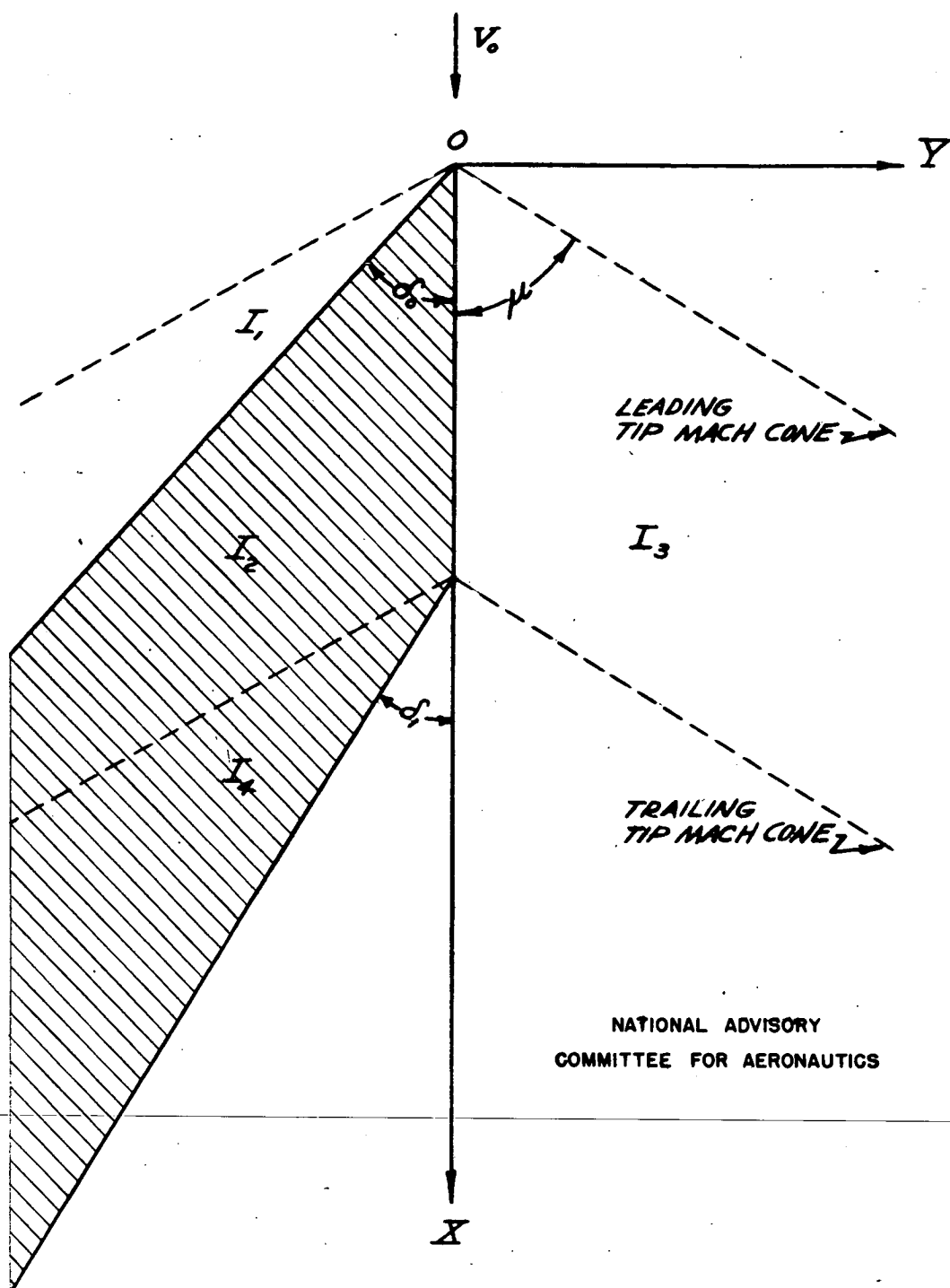


FIGURE 5.— TIP OF SWEEPED FORWARD LIFTING SURFACE WITH TRACES OF MACH CONES, COORDINATE SYSTEM, AND REGIONS DEFINED FOR FORMULAS 33, 34, 36, AND 41.

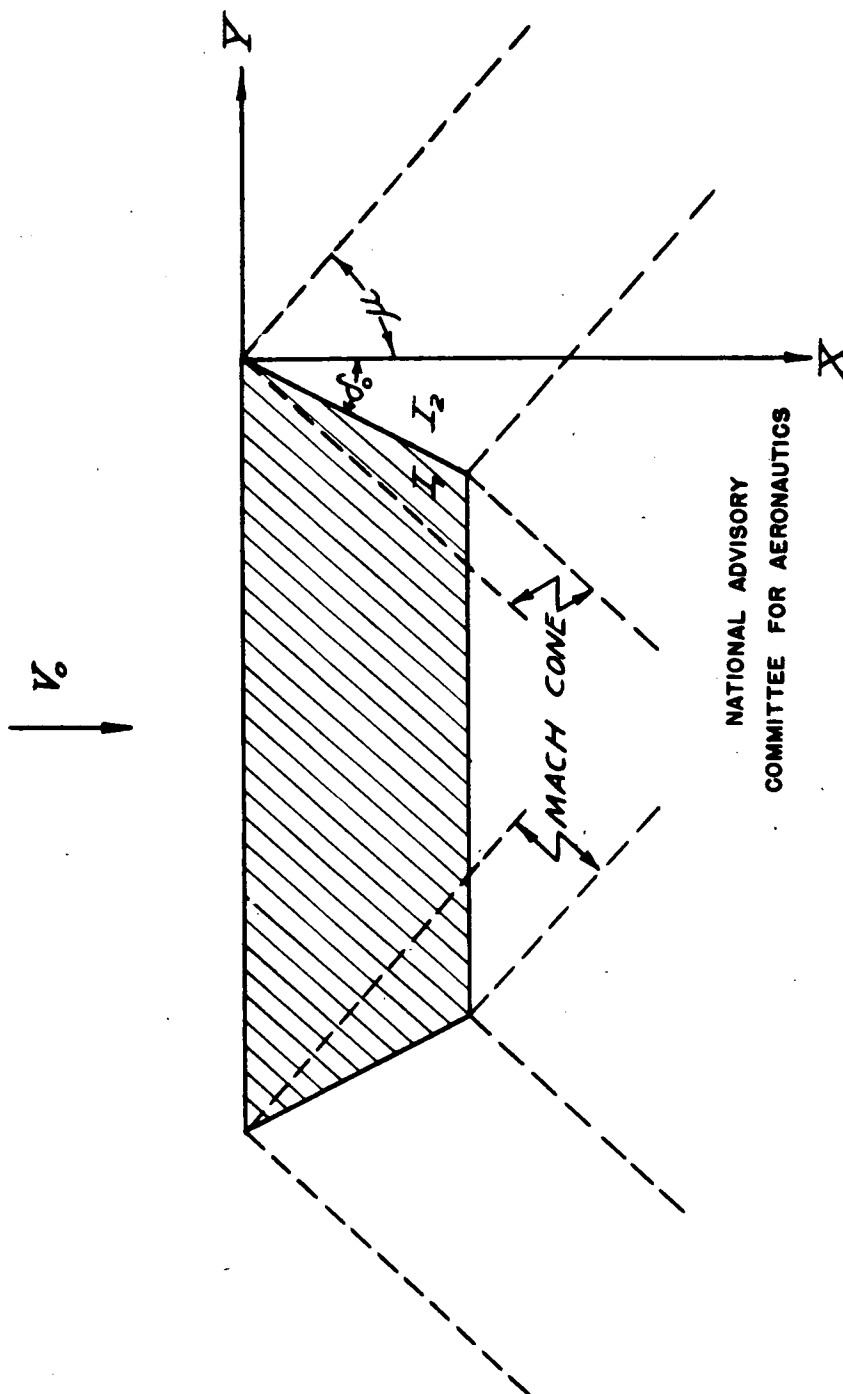


FIGURE 6. — TRAPEZOIDAL LIFTING SURFACE
WITH TRACES OF MACH CONES, COORDINATE
SYSTEM, AND REGIONS DEFINED FOR
FORMULAS 42 AND 43.

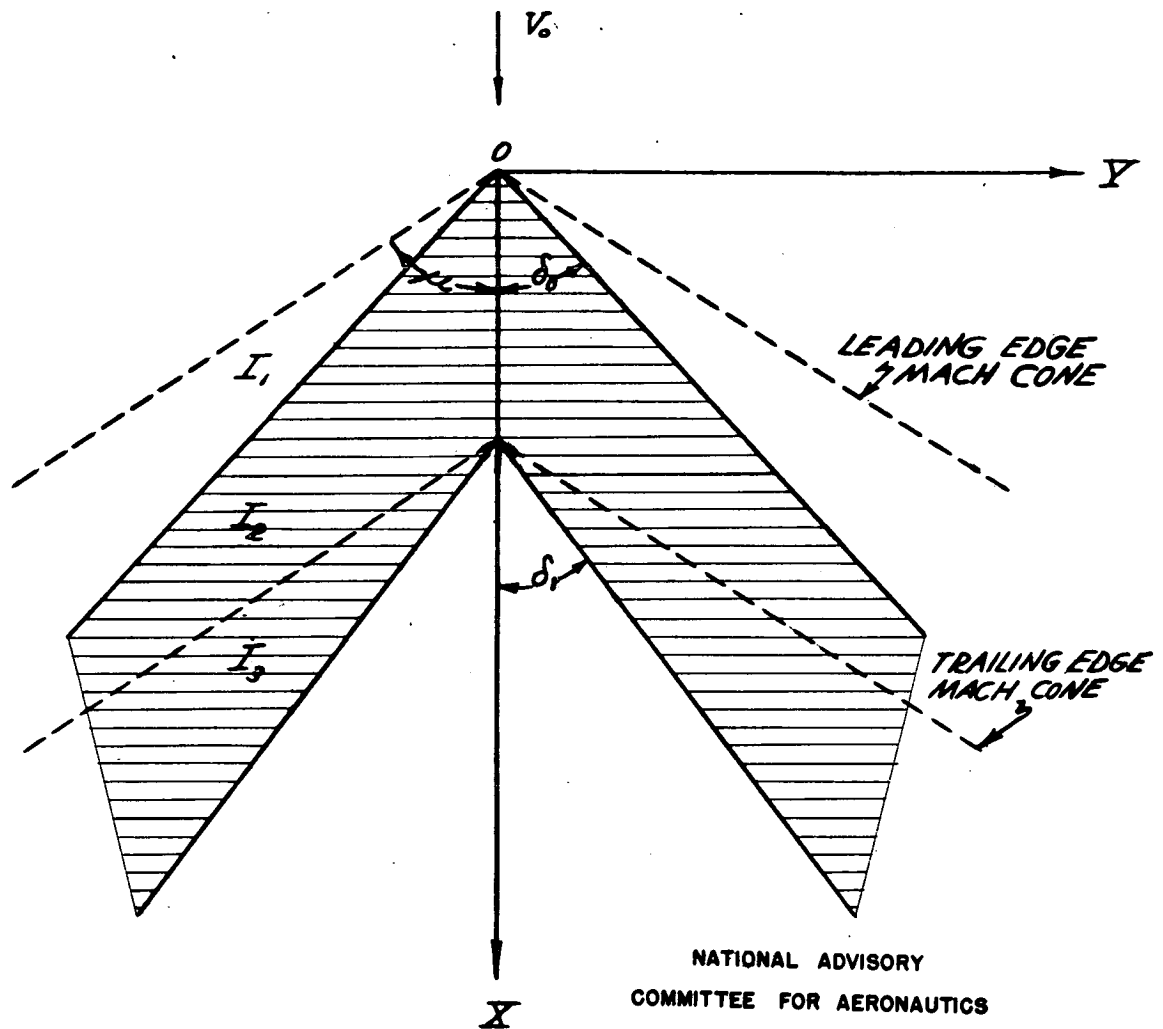


FIGURE 7. — SWEEPED-BACK LIFTING SURFACE
WITH TRACES OF MACH CONES, COORDINATE
SYSTEM, AND REGIONS DEFINED FOR FORMULAS
44, 45, AND 47.

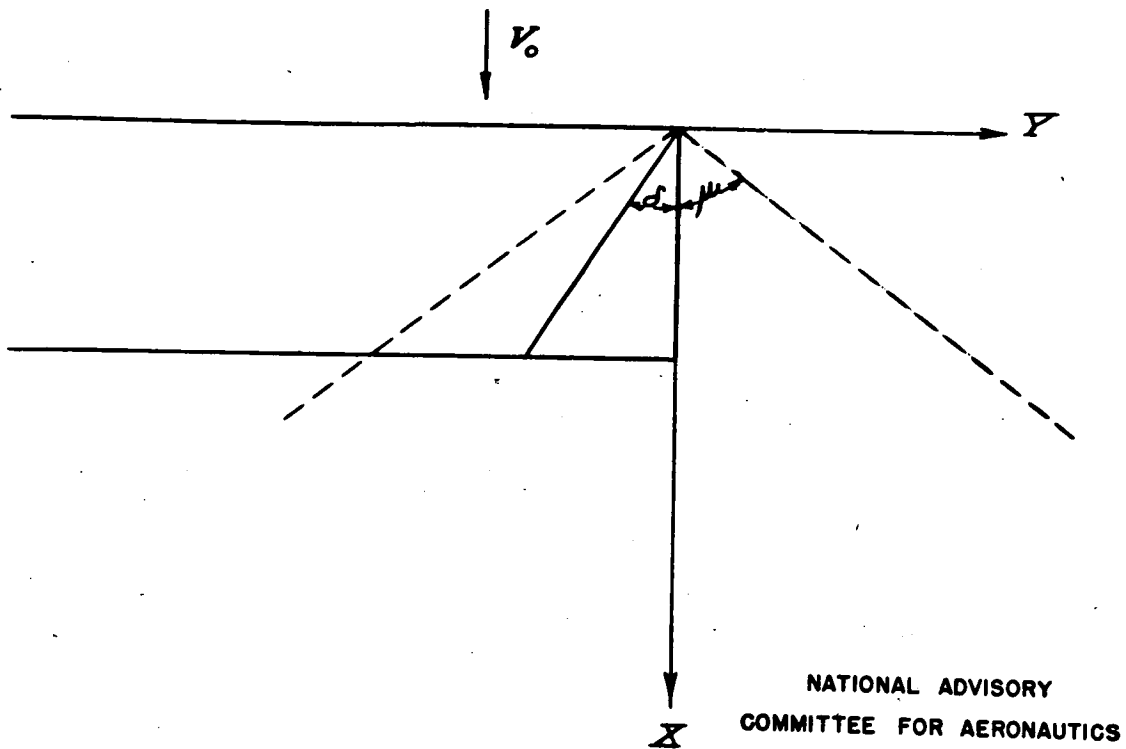


FIGURE 8. — RECTANGULAR PLANFORM BUILT OF
SUPERIMPOSED TRAPEZOIDAL LIFTING SURFACES
WITH VARIABLE RAKE.

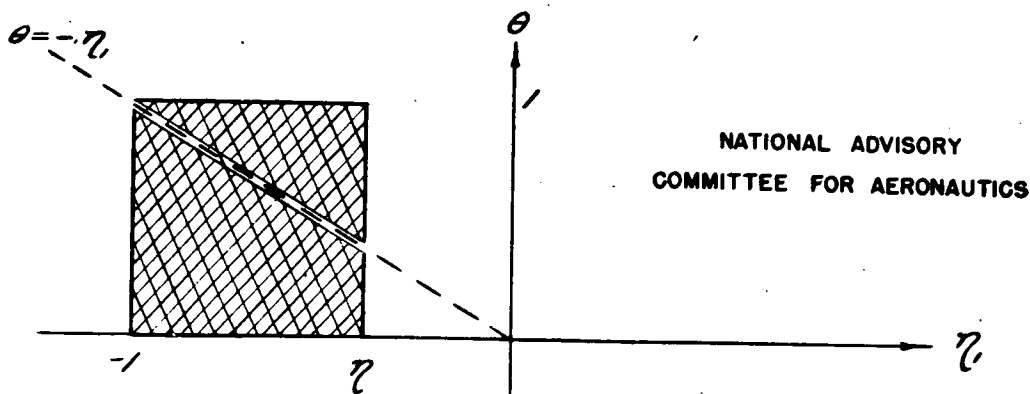


FIGURE 9. — REGION OF INTEGRATION SHOWING
LINE OF SINGULARITY FOR EQUATION (54).

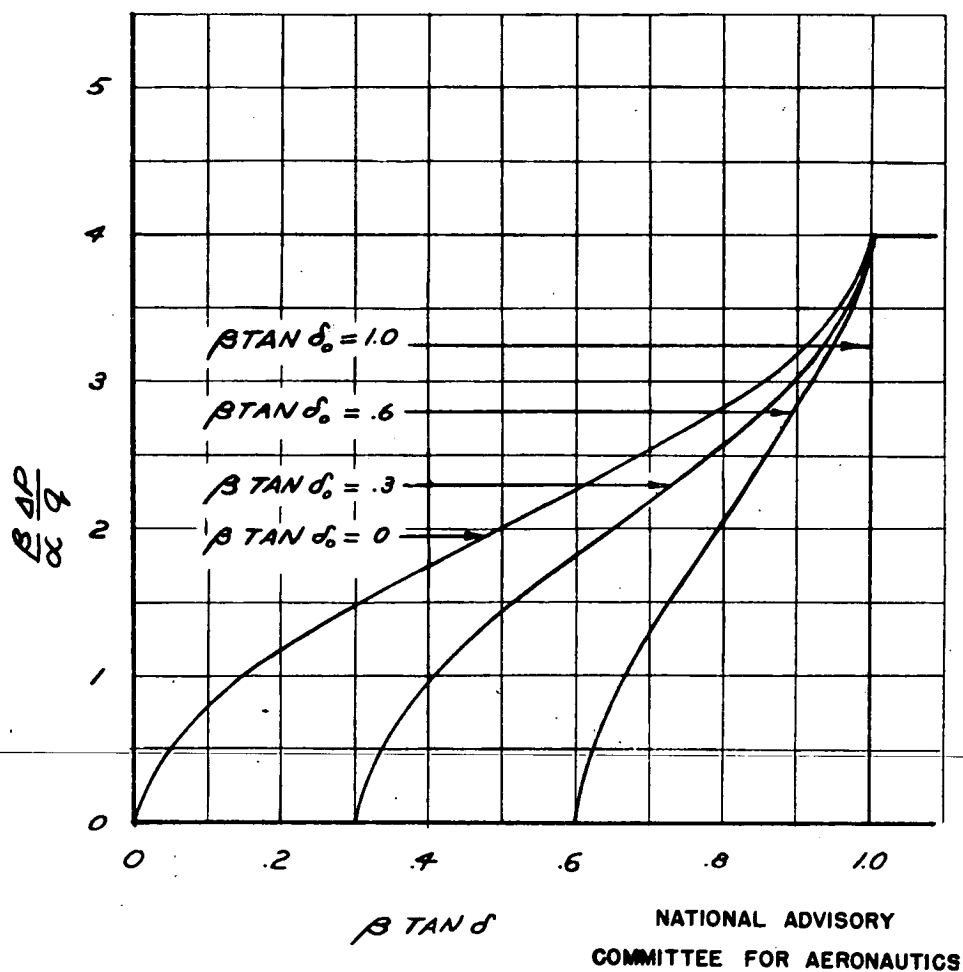
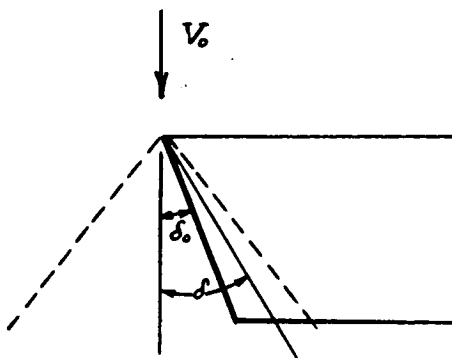


FIGURE 10.- LOAD DISTRIBUTION OVER TIP FOR VARIOUS TRAPEZOIDAL PLANFORMS.

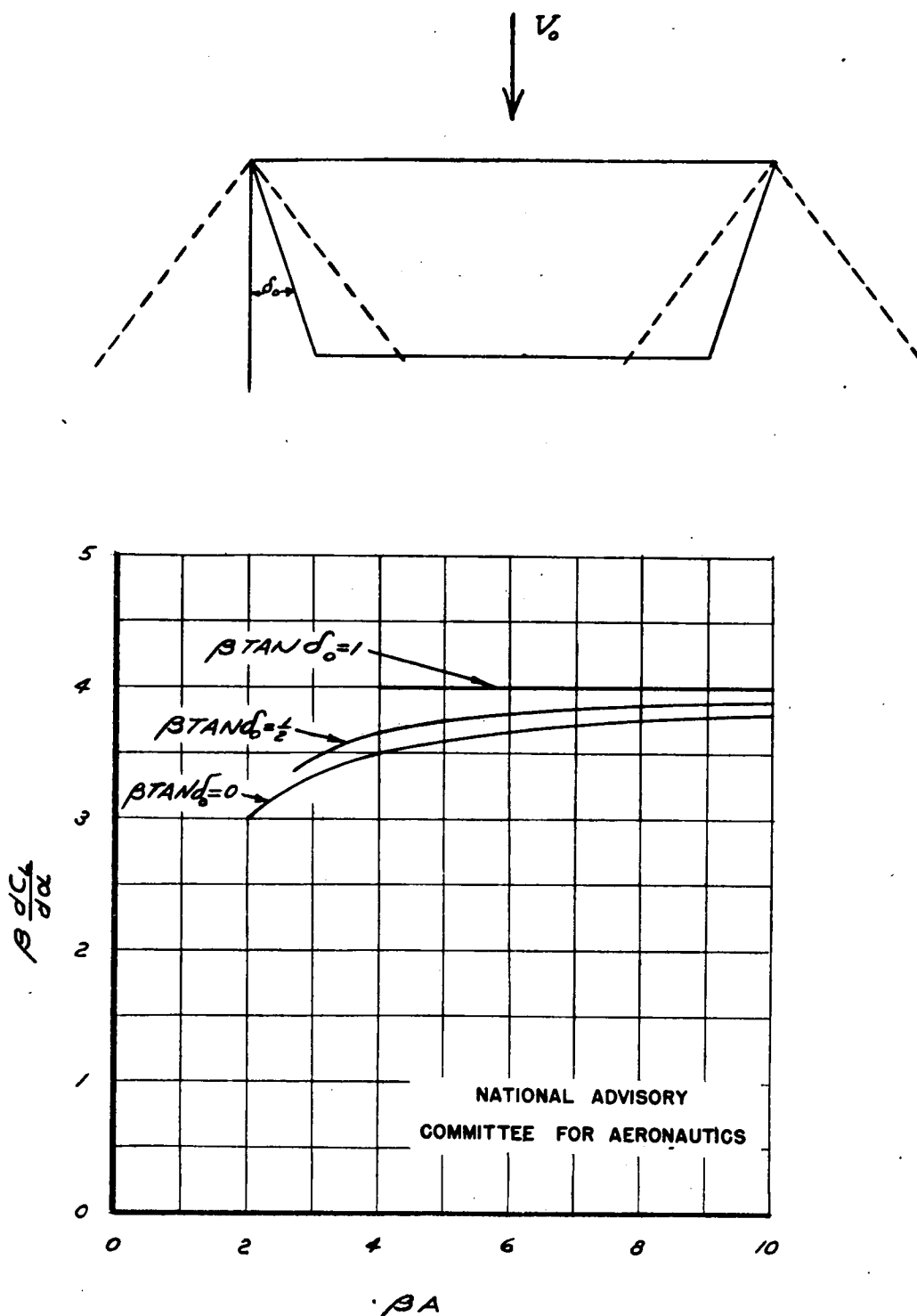
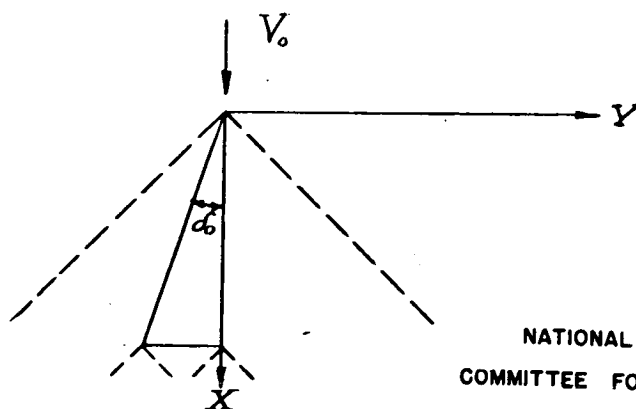
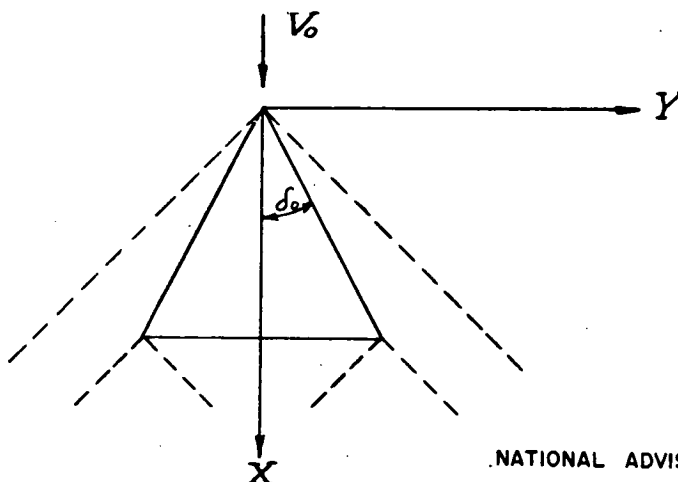


FIGURE 11. — VARIATION OF REDUCED LIFT-CURVE SLOPE $\beta \frac{dC_L}{d\alpha}$ WITH REDUCED ASPECT RATIO βA FOR VARIOUS TRAPEZOIDAL PLANFORMS.



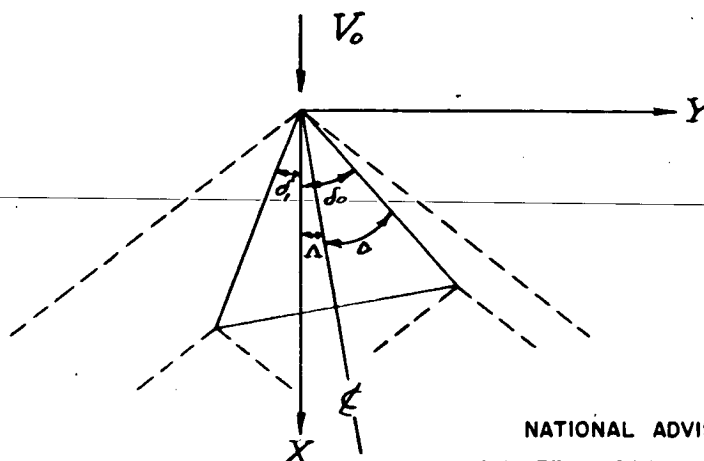
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FIGURE 12.- TRIANGULAR FLAT PLATE LIFTING SURFACES.
(a) TYPE 1.



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FIGURE 12(b).- TYPE 2.



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FIGURE 12(c).- TYPE 3.

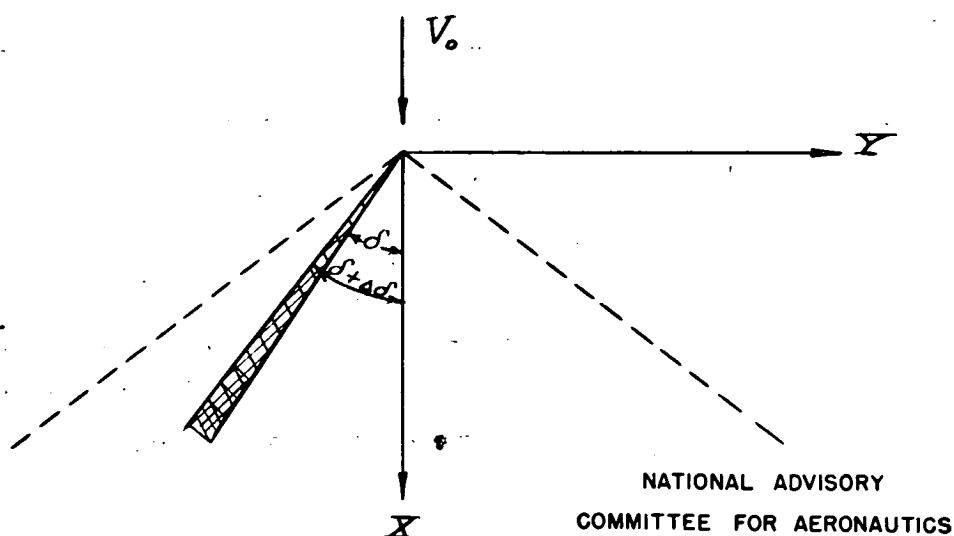


FIGURE 13.— ELEMENTAL LIFTING SURFACE OF CONSTANT LOAD.

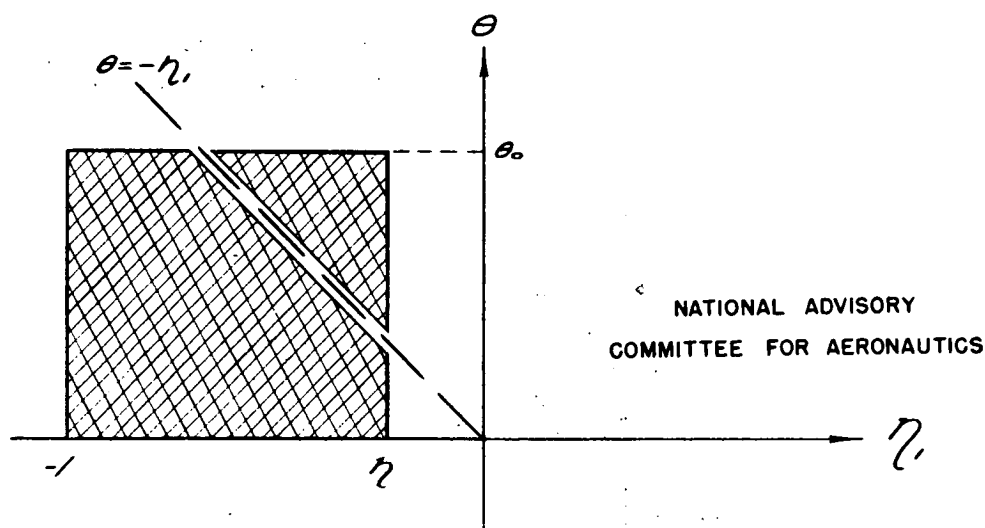


FIGURE 14.— REGION OF INTEGRATION SHOWING LINE OF SINGULARITY FOR EQUATIONS (80) AND (94)

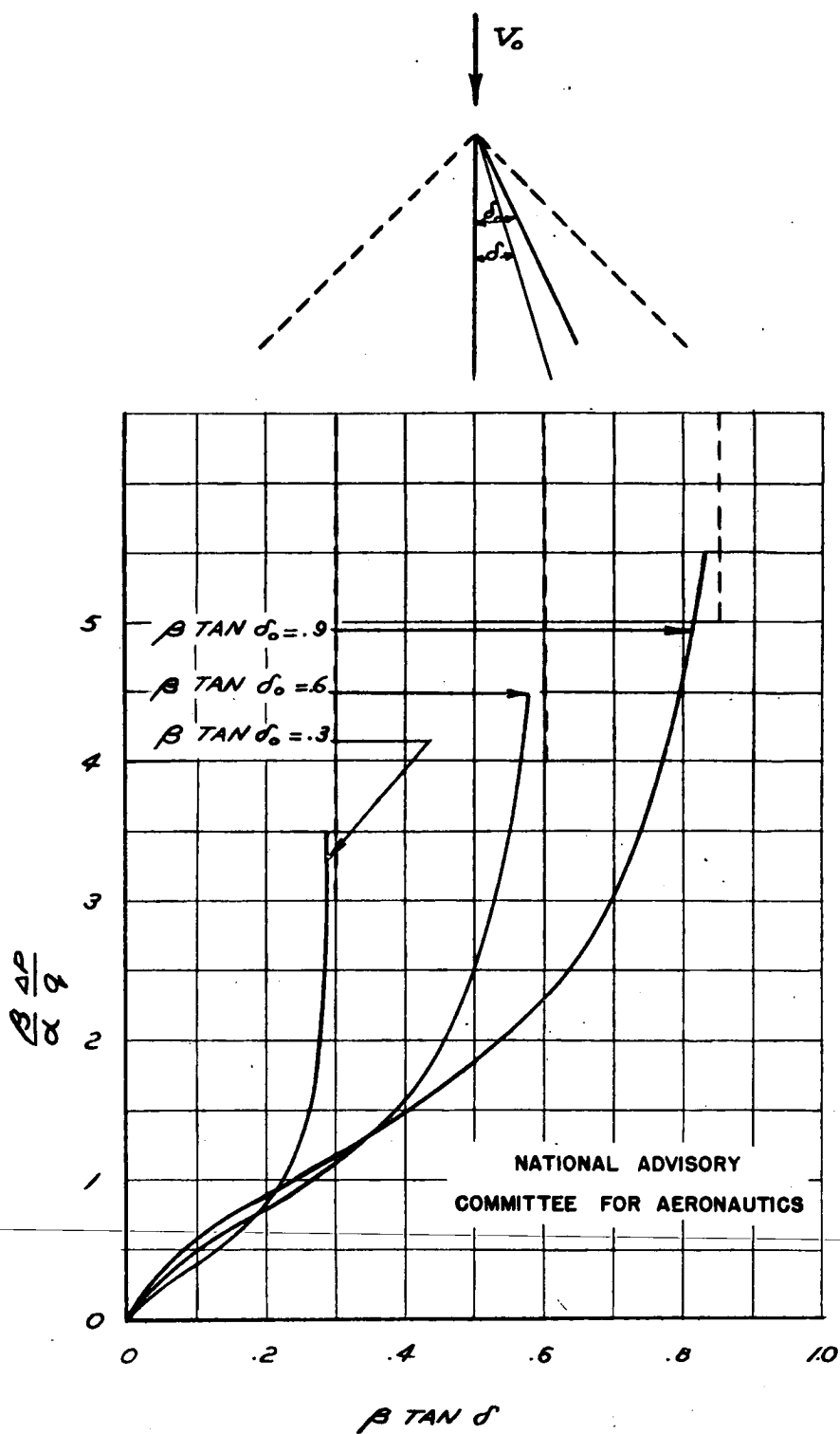


FIGURE 15.—LOAD DISTRIBUTION OVER TRIANGULAR PLANFORMS OF TYPE I.

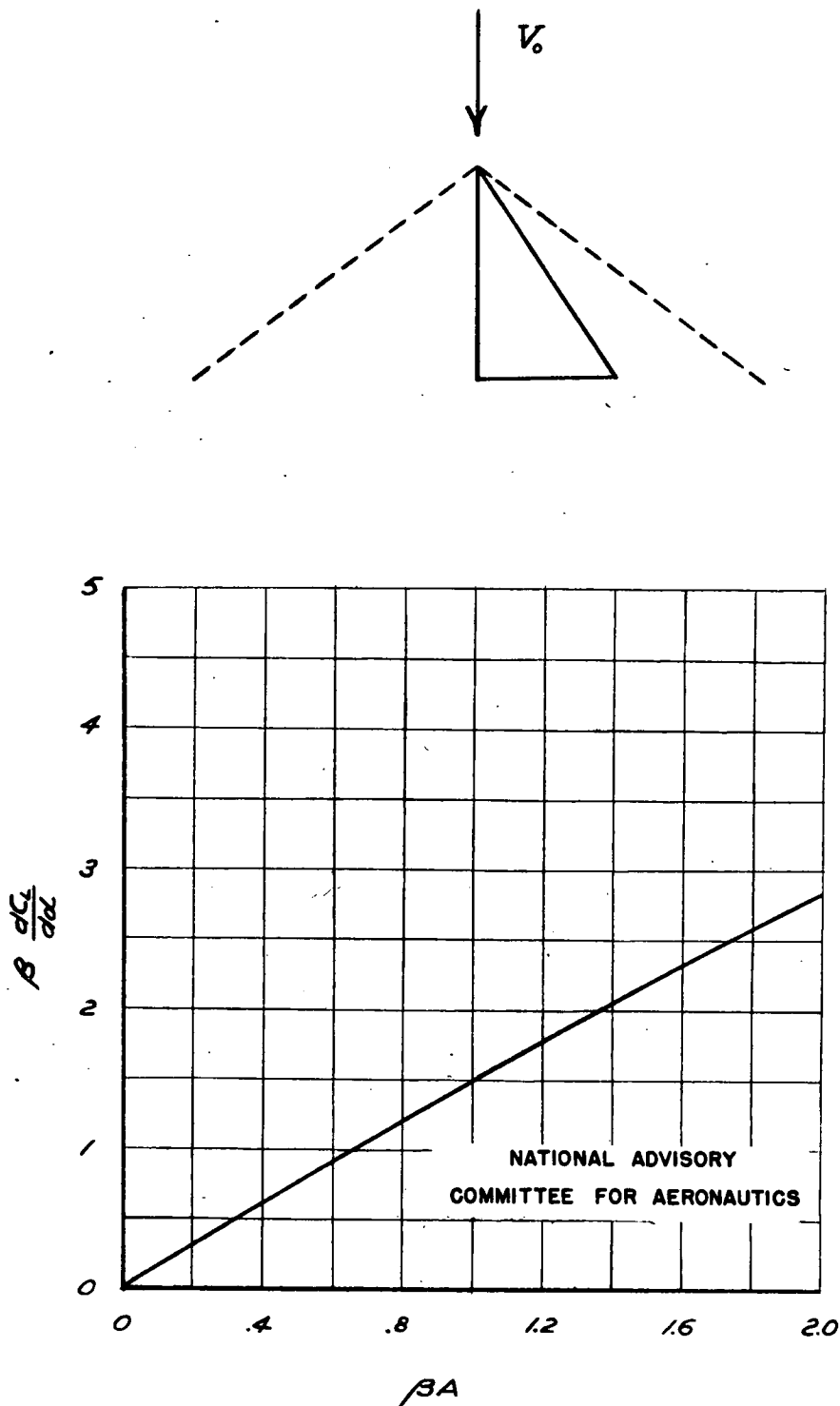


FIGURE 16.— VARIATION OF REDUCED LIFT-CURVE SLOPE $\beta \frac{dC_L}{d\alpha}$ WITH REDUCED ASPECT RATIO βA FOR PLANFORMS OF TYPE I.

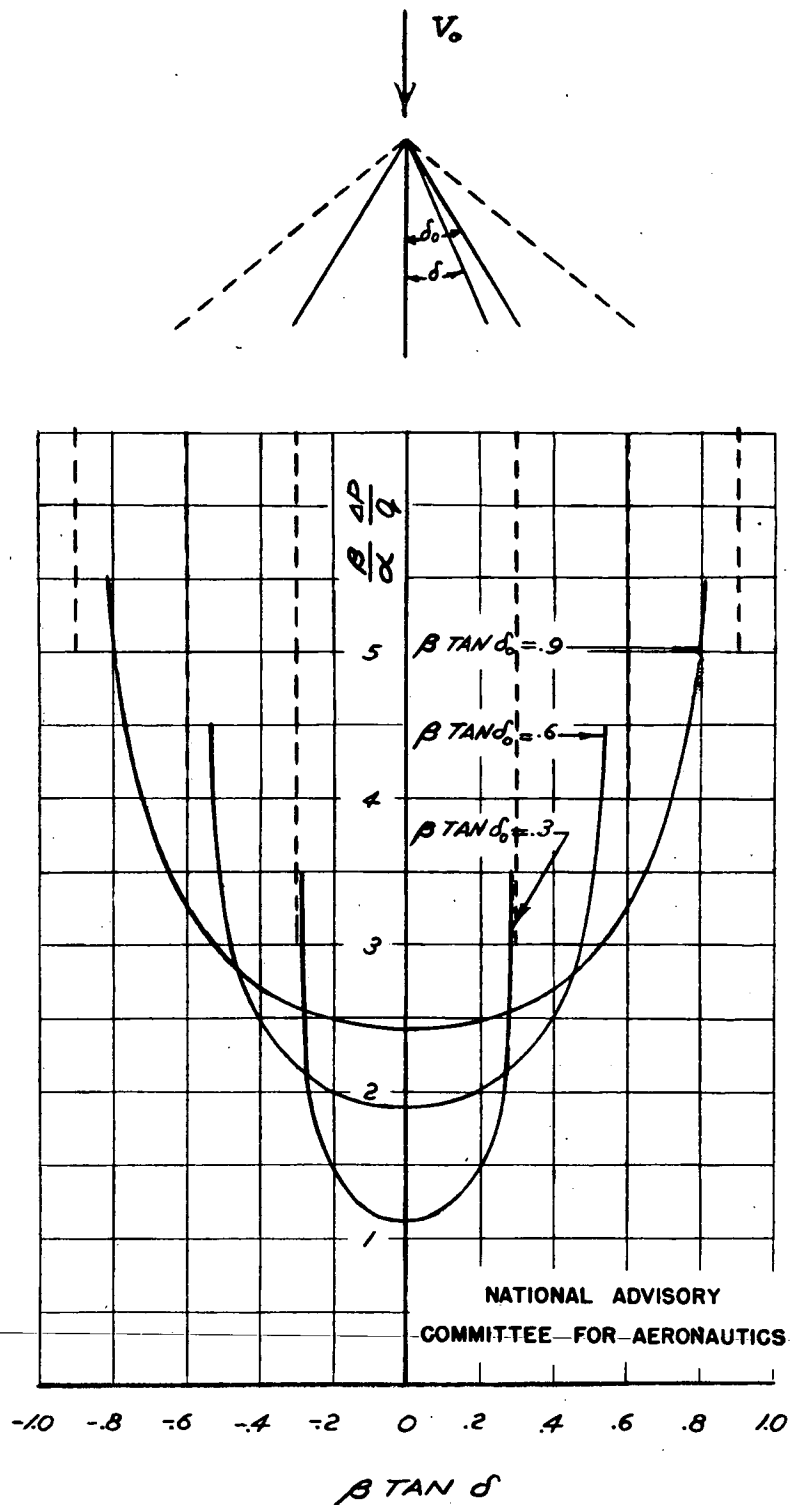


FIGURE 17.—LOAD DISTRIBUTION OVER TRIANGULAR PLANFORMS OF TYPE 2.

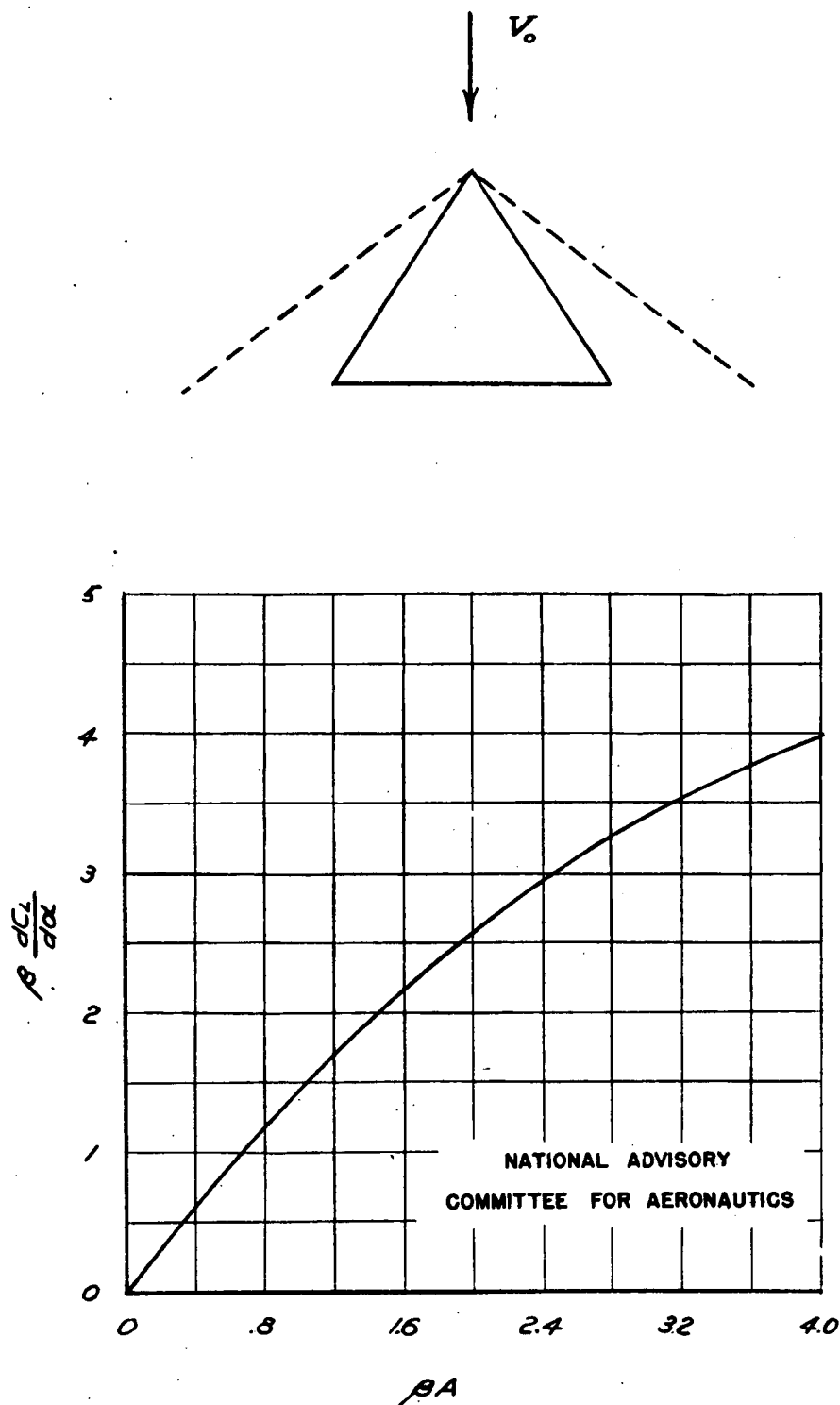


FIGURE 18.— VARIATION OF REDUCED LIFT-CURVE SLOPE $\beta \frac{dC_L}{d\alpha}$ WITH REDUCED ASPECT RATIO βA FOR PLANFORMS OF TYPE 2.

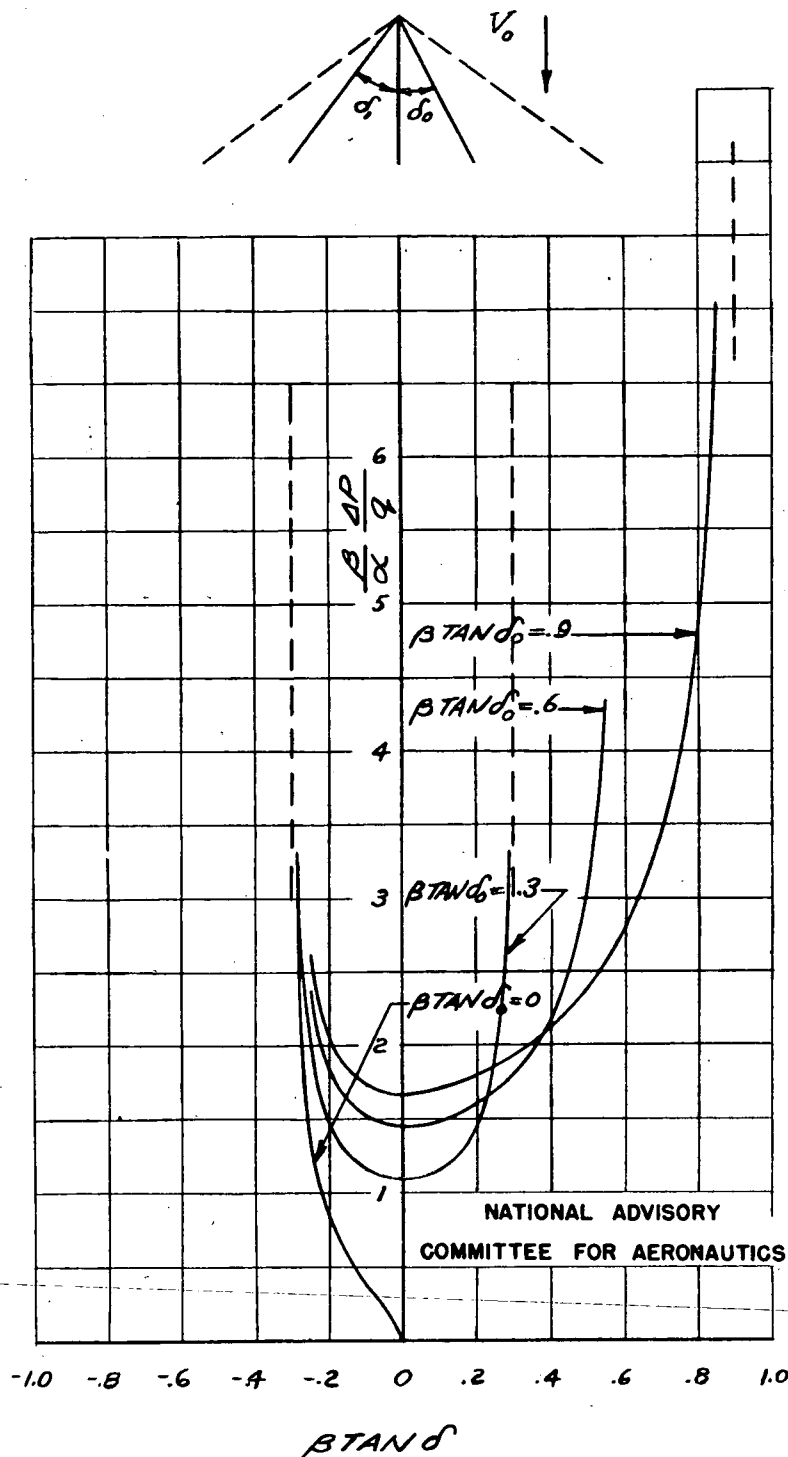
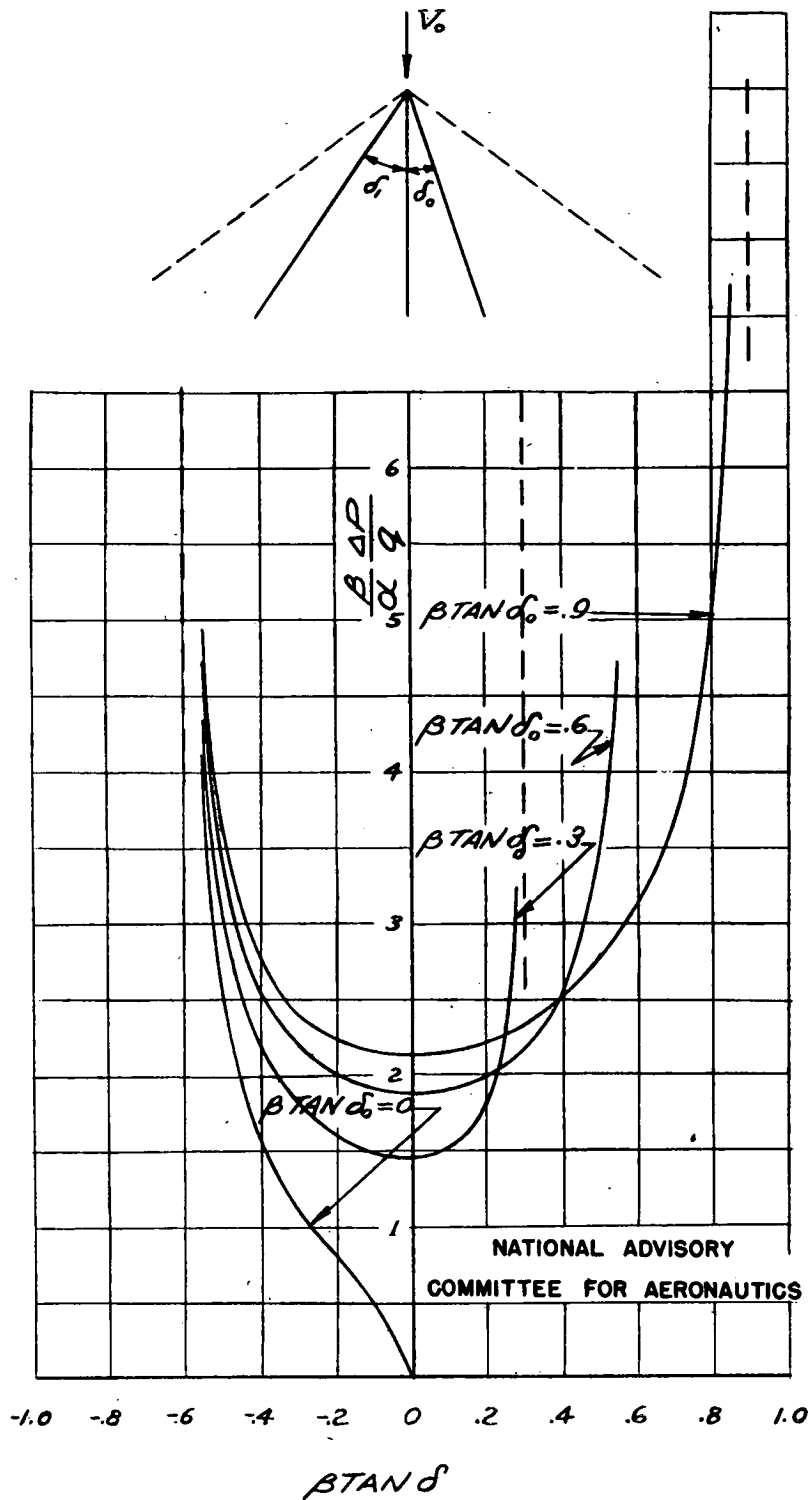


FIGURE 19.—LOAD DISTRIBUTION OVER TRI-
ANGULAR PLANFORMS OF TYPE 3.
(a) $\beta \tan \delta_0 = .3$.

FIGURE 19. - CONTINUED. (b) $\beta \tan \delta_0 = .6$

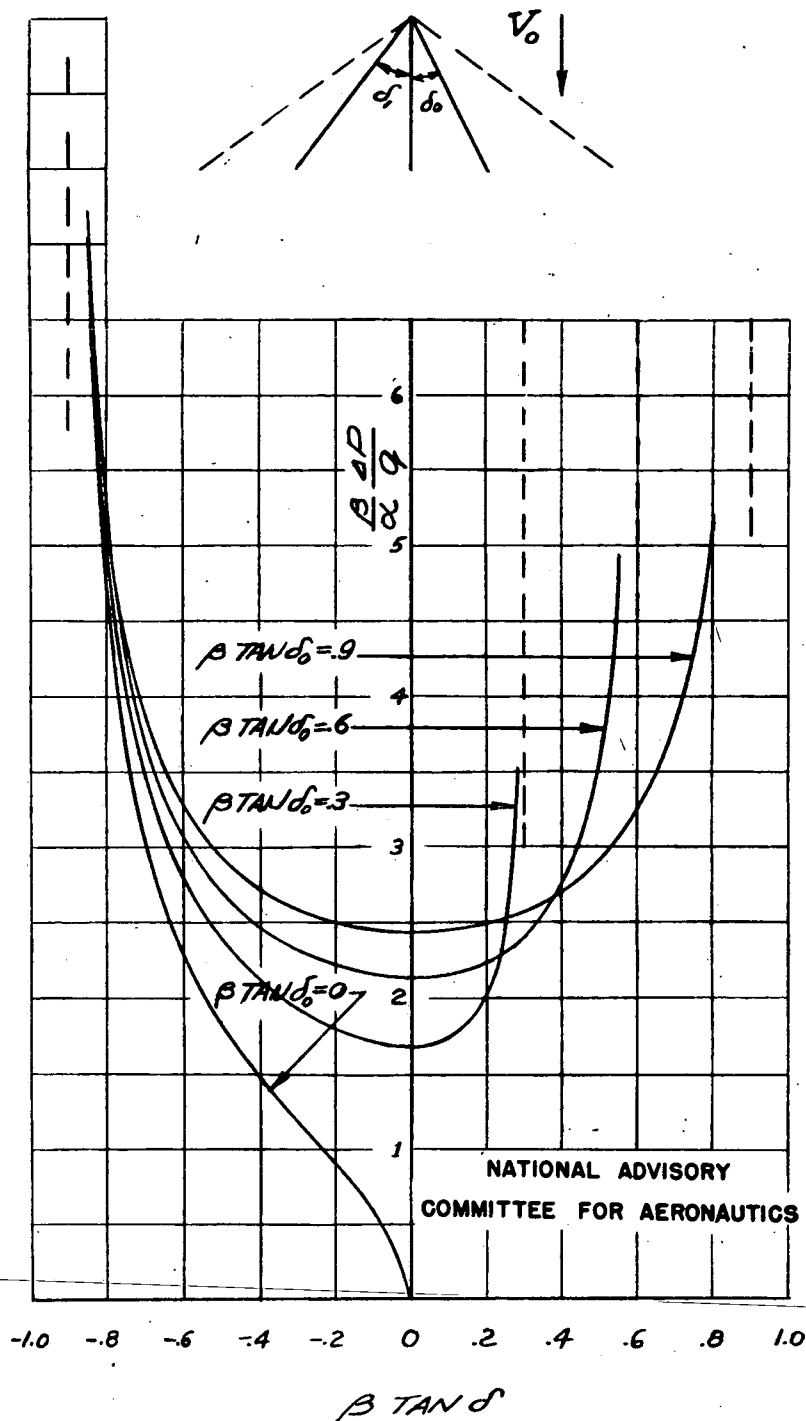


FIGURE 19. - CONCLUDED. (C) $\beta \tan \delta_0 = .9$.

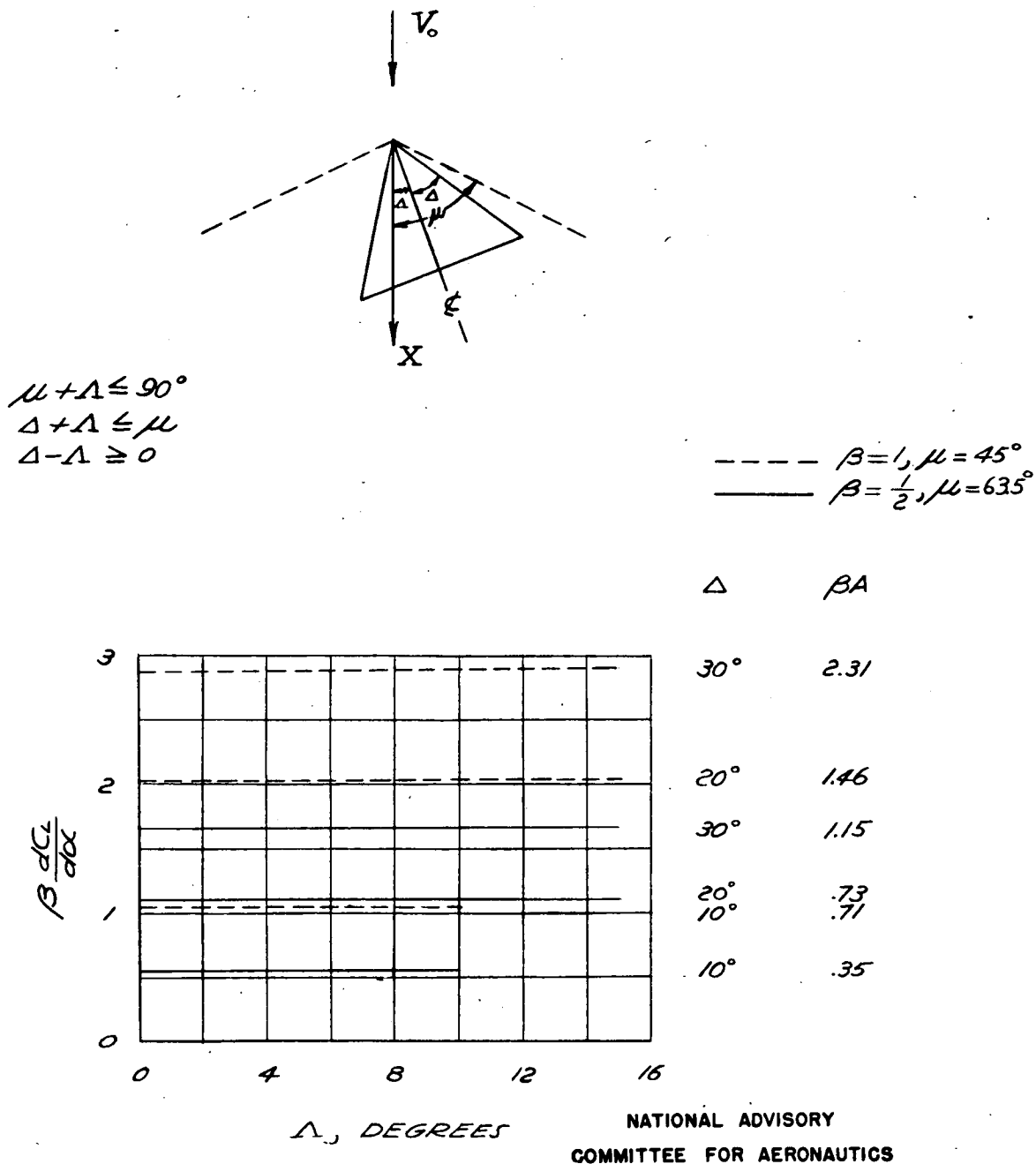


FIGURE 20.— VARIATION OF REDUCED LIFT-CURVE SLOPE $\beta \frac{dC_L}{d\alpha}$ WITH ANGLE OF SIDESLIP Δ FOR SOME PLANFORMS OF TYPE 3.